United Kingdom
Mathematics Trust

Senior Mathematical Challenge

Organised by the United Kingdom Mathematics Trust

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Solutions and investigations

November 2020

These solutions augment the shorter solutions also available online. The shorter solutions sometimes leave out details. The solutions given here are full solutions, as explained below. In some cases alternative solutions are given. There are also many additional problems for further investigation. We welcome comments on these solutions and the additional problems. Please send them to enquiry@ukmt.org.uk.

The Senior Mathematical Challenge (SMC) is a multiple-choice paper. Sometimes you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the SMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the questions without any alternative answers. So we aim at including for each question a complete solution with each step explained (or, occasionally, left as an exercise), and not based on the assumption that one of the given alternatives is correct. We hope that these solutions provide a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the British Mathematical Olympiad, the Mathematical Olympiad for Girls and similar competitions).

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
D	C	A	B	C	E	D	D	B	C	E	B	D	B	A	E	A	E	A	E	C	A	D	B	C

1. What is the value of $\dfrac{2020}{20 \times 20}$?

A 10.1 B 5.5 C 5.1 D 5.05 E 0.55

SOLUTION **D**

Note: In the absence of a calculator, the best way to tackle this question is to first do some cancelling and then the division. There is more than one way to do this. In our method we twice divide the numerator and denominator by 10.

$$\frac{2020}{20 \times 20} = \frac{202}{2 \times 20} = \frac{20.2}{2 \times 2} = \frac{20.2}{4} = 5.05.$$

FOR INVESTIGATION

1.1 What is the value of the following?

(a) $\dfrac{2020}{2 \times 2}$,

(b) $\dfrac{2020}{200 \times 200}$,

(c) $\dfrac{2020}{2 \times 200}$.

1.2 Find the positive integer n such that $\dfrac{2020}{n \times n} = 126.25$.

1.3 Find all the positive integers n for which $\dfrac{2020}{n \times n}$ is an integer.

2. What is the remainder when 1234×5678 is divided by 5?

A 0 B 1 C 2 D 3 E 4

SOLUTION **C**

The units digit of 1234×5678 is the same as the units digit of 4×8, and hence is 2. Therefore the remainder when 1234×5678 is divided by 5 is 2.

FOR INVESTIGATION

2.1 What is the remainder when 1234×5678 is divided by 3?

2.2 What is the remainder when 1234×5678 is divided by 11?

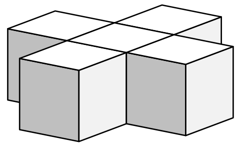
2.3 The integer m has remainder 3 when it is divided by 11. The integer n has remainder 4 when divided by 11.

What is the remainder when mn is divided by 11?

3. A shape is made from five unit cubes, as shown.

What is the surface area of the shape?

A 22 B 24 C 26 D 28 E 30



Solution **A**

The surface of the shape is made up of two faces of the central cube, and five faces of each of the four other cubes.

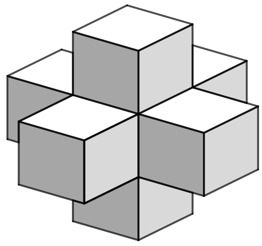
Therefore the surface is made up of $2 + 4 \times 5 = 22$ square faces each of size 1×1.

Hence the surface area of the shape is 22.

For investigation

3.1 The shape shown is made by adding two unit cubes to the shape of this question. It is made from seven unit cubes.

What is the surface area of the shape?



4. The numbers p, q, r and s satisfy the equations $p = 2$, $p \times q = 20$, $p \times q \times r = 202$ and $p \times q \times r \times s = 2020$.

What is the value of $p + q + r + s$?

A 32 B 32.1 C 33 D 33.1 E 34

Solution **B**

We have

$$\begin{aligned} p &= 2, \\ q &= \frac{p \times q}{p} = \frac{20}{2} = 10, \\ r &= \frac{p \times q \times r}{p \times q} = \frac{202}{20} = 10.1, \end{aligned}$$

and

$$s = \frac{p \times q \times r \times s}{p \times q \times r} = \frac{2020}{202} = 10.$$

Therefore $p + q + r + s = 2 + 10 + 10.1 + 10 = 32.1$.

5. What is $\sqrt{123454321}$?

A 1111111 B 111111 C 11111 D 1111 E 111

SOLUTION **C**

Note: You could not be expected to be able to calculate the value of $\sqrt{123454321}$ without the use of a calculator. So you need to find some other way to select the correct option. We use a method based on the size of the number 123454321.

We have

$$10^8 < 123454321 < 10^{10}.$$

Therefore

$$\sqrt{10^8} < \sqrt{123454321} < \sqrt{10^{10}},$$

that is,

$$10^4 < \sqrt{123454321} < 10^5.$$

The correct option is therefore the only one that is between 10^4 and 10^5. Therefore, of the given options, it is 11111 that equals $\sqrt{123454321}$.

FOR INVESTIGATION

5.1 The answer given above assumes that one of the given options is correct.

Verify that $11111 = \sqrt{123454321}$ by checking that $11111^2 = 123454321$.

6. There are fewer than 30 students in the A-level mathematics class. One half of them play the piano, one quarter play hockey and one seventh are in the school play.

How many of the students play hockey?

A 3 B 4 C 5 D 6 E 7

SOLUTION **E**

Because one half of the students play the piano, the number of students is a multiple of 2.

Because one quarter of the students play hockey, the number of students is a multiple of 4.

Because one seventh of the students are in the school play, the number of students is a multiple of 7.

Therefore the number of students is a multiple of 2, 4 and 7. Hence the number of students is a multiple of 28.

Because there are fewer than 30 students in the class, it follows that there are 28 students in the class.

Therefore, because one quarter of the 28 students play hockey, the number of students who play hockey is 7.

7. Official UK accident statistics showed that there were 225 accidents involving teapots in one year. However, in the following year there were 47 such accidents.

What was the approximate percentage reduction in recorded accidents involving teapots from the first year to the second?

A 50 B 60 C 70 D 80 E 90

SOLUTION **D**

The reduction in the number of teapot accidents in the second year was $225 - 47 = 178$.

178 as a percentage of 225 is

$$\frac{178}{225} \times 100 \approx \frac{180}{225} \times 100 = \frac{20}{25} \times 100 = 20 \times 4 = 80.$$

8. What is the largest prime factor of $106^2 - 15^2$?

A 3 B 7 C 11 D 13 E 17

SOLUTION **D**

Note: It is not a good idea to attempt to calculate 106^2 and 15^2, then do a subtraction and finally attempt to factorize the resulting answer.

Instead we make use of the standard factorization of the difference of two squares:

$$x^2 - y^2 = (x - y)(x + y).$$

We have

$$\begin{aligned} 106^2 - 15^2 &= (106 - 15)(106 + 15) \\ &= 91 \times 121 \\ &= 7 \times 13 \times 11 \times 11. \end{aligned}$$

Therefore the prime factorization of $106^2 - 15^2$ is $7 \times 11^2 \times 13$, from which we see that its largest prime factor is 13.

FOR INVESTIGATION

8.1 What is the largest prime factor of $300^2 - 3^2$?

9. In 2018, a racing driver was allowed to use the Drag Reduction System provided that the car was within 1 second of the car ahead. Suppose that two cars were 1 second apart, each travelling at 180 km/h (in the same direction!).

How many metres apart were they?

A 100 B 50 C 10 D 5 E 1

Solution **B**

The distance apart of the cars was the distance that a car travelling at 180 km/h travels in 1 second.

There are 1000 metres in one kilometre. Hence there are 180×1000 metres in 180 km.

There are 60×60 seconds in each hour.

It follows that at 180 km/h a car travels 180×1000 metres in 60×60 seconds.

Therefore the number of metres that it travels in 1 second is

$$\frac{180 \times 1000}{60 \times 60} = \frac{3 \times 1000}{60} = \frac{1000}{20} = 50.$$

Therefore when the cars are 1 second apart, they are 50 metres apart.

For investigation

9.1 The Highway Code gives the following table of typical *stopping distances* in metres for motor vehicles travelling at different velocities.

velocity	stopping distance
32 km/h	12 m
48 km/h	23 m
64 km/h	36 m
80 km/h	53 m
96 km/h	73 m
112 km/h	96 m

For each velocity, how far apart in seconds should two cars travelling in the same direction at that velocity be so that their distance apart is the same as the corresponding stopping distance given in the above table?

9.2 *Note*: The Highway Code adds that "The distances shown are a general guide. The distance will depend on your attention (thinking distance), the road surface, the weather conditions and the condition of your vehicle at the time. "

The Highway Code gives a thinking distance of 12 m for a car travelling at 64 km/h. How much thinking time does that correspond to?

10. Six friends Pat, Qasim, Roman, Sam, Tara and Uma, stand in a line for a photograph. There are three people standing between Pat and Qasim, two between Qasim and Roman and one between Roman and Sam. Sam is not at either end of the line.

How many people are standing between Tara and Uma?

A 4 B 3 C 2 D 1 E 0

SOLUTION **C**

We indicate each of the friends by the first letter of their name, and a person whose name we are not yet sure about by an asterisk (*).

We can assume, without loss of generality, that, from the point of view of the photographer, Qasim is to the right of Pat. Because there are three people standing between Pat and Qasim, the line is either

$$* \; P \; * \; * \; * \; Q \quad \text{or} \quad P \; * \; * \; * \; Q \; * .$$

There are two people between Qasim and Roman. Roman cannot be to the right of Quasim, because there is at most one friend to the right of Quasim. Therefore Roman is to the left of Quasim and the line is either

$$* \; P \; R \; * \; * \; Q \quad \text{or} \quad P \; R \; * \; * \; Q \; * .$$

There is one person between Roman and Sam. Therefore either Sam is immediately to the left of Pat, or immediately to the left of Qasim.

If Sam is immediately to the left of Pat, the line would be

$$S \; P \; R \; * \; * \; Q.$$

However this is impossible because Sam is not at either end of the line.

Therefore Sam is immediately to the left of Qasim. Hence the line is either

$$* \; P \; R \; * \; S \; Q \quad \text{or} \quad P \; R \; * \; S \; Q \; *$$

with Tara and Uma occupying the two places marked by asterisks.

We see that, however Tara and Uma occupy these two places, the number of people standing between Tara and Uma will be 2.

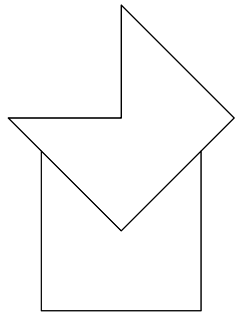
FOR INVESTIGATION

10.1 Specify one additional piece of information that would make it possible to work out the exact order of the six friends from left to right, as seen by the photographer.

11. Two congruent pentagons are each formed by removing a right-angled isosceles triangle from a square of side-length 1. The two pentagons are then fitted together as shown.

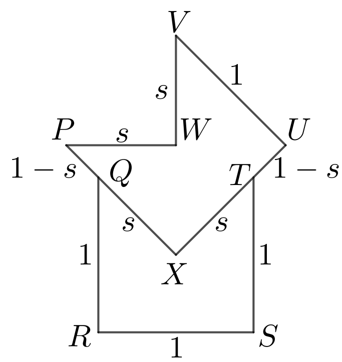
What is the length of the perimeter of the octagon formed?

A 4   B $4 + 2\sqrt{2}$ C 5   D $6 - 2\sqrt{2}$ E 6



SOLUTION **E**

Let P, Q, R, S, T, U, V and W be the vertices of the octagon, as shown in the diagram, and let X be the vertex as shown.



The sides PX, XU, UV, QR, RS and ST all have length 1.

Let PW have length s. Then, because PWV is a right-angled isosceles triangle, VW also has length s.

Because the pentagons $PXUVW$ and $QRSTX$ are congruent, QX and TX also have length s.

It follows that PQ and TU each have length $1 - s$.

We can now deduce the the length of the perimeter of the octagon $PQRSTUVW$ is

$$(1 - s) + 1 + 1 + 1 + (1 - s) + 1 + s + s = 6.$$

COMMENTARY

Note that in order to work out the length of the perimeter we did not need to know the value of s. It is, however, not difficult to find the value of s. You are asked to do this in Problem 11.1.

FOR INVESTIGATION

11.1 Find the length of QX.

11.2 Find the area of the octagon $PQRSTUVW$.

12. A three-piece suit consists of a jacket, a pair of trousers and a waistcoat. Two jackets and three pairs of trousers cost £380. A pair of trousers costs the same as two waistcoats.

What is the cost of a three-piece suit?

A £150 B £190 C £200 D £228 E more information needed

Solution **B**

We let the cost of a jacket, a pair of trousers and a waistcoat be £J, £T and £W, respectively.

From the information given in the question we can deduce that

$$2J + 3T = 380 \qquad (1)$$

and

$$T = 2W \qquad (2)$$

Note: In this problem we have three unknowns J, T and W, but only two equations. We don't have enough information to enable us to find the values of J, T and W. However, we can deduce the value of $J + T + W$ which is what we need to know. We give two methods for doing this.

Method 1

If we subtract equation (2) from equation (1) we obtain

$$2J + 2T = 380 - 2W.$$

Hence

$$2J + 2T + 2W = 380.$$

Hence, dividing both sides of this last equation by 2, we obtain

$$J + T + W = 190.$$

Therefore the cost of a three piece suit is £190.

Method 2

It follows from equation (2) that $W = \frac{1}{2}T$. Hence,

$$\begin{aligned} J + T + W &= J + T + \tfrac{1}{2}T \\ &= J + \tfrac{3}{2}T \\ &= \tfrac{1}{2}(2J + 3T) \\ &= \tfrac{1}{2}(380), \qquad \text{by equation (1),} \\ &= 190. \end{aligned}$$

Therefore the cost of a three-piece suit is £190.

13. The number $16! \div 2^k$ is an odd integer. Note that $n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$.

What is the value of k?

A 9 B 11 C 13 D 15 E 17

SOLUTION **D**

The number k that we seek is such that $16! = 2^k \times q$, where q is an odd integer. Thus k is the power of 2 that occurs in the prime factorization of 16!.

METHOD 1

To find the highest power of 2 that is a factor of 16! we need only consider the even numbers that occur in the product $1 \times 2 \times \cdots \times 15 \times 16$ that gives the value of 16!

In the following table we give the powers of 2 that divide each of these factors.

factor	2	4	6	8	10	12	14	16
power of 2	1	2	1	3	1	2	1	4

The power of 2 in the prime factorization of 16! is the sum of the powers in the second row of this table. That is, it is $1 + 2 + 1 + 3 + 1 + 2 + 1 + 4 = 15$.

Hence, the required value of k is 15.

METHOD 2

Note:
In this method we use the formula, usually attributed to the French mathematician Adrian-Marie Legendre (1752-1833), for the highest power of a prime p that divides $n!$. This formula makes use of the *floor* function which we first explain.

We define the *floor* of x, written as $\lfloor x \rfloor$, to be the largest integer that is not larger than x.

For example, $\lfloor 2\frac{6}{7} \rfloor = 2$, $\lfloor 4.275 \rfloor = 4$, $\lfloor \pi \rfloor = 3$, $\lfloor 7 \rfloor = 7$, $\lfloor 0.43 \rfloor = 0$ and $\lfloor -5.23 \rfloor = -6$.

According to Legendre's formula, the highest power of the prime p that divides $n!$ is given by

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \ldots .$$

At first sight, the formula involves an infinite sum. However, if $p^k > n$, we have $0 < \dfrac{n}{p^k} < 1$ and hence $\left\lfloor \dfrac{n}{p^k} \right\rfloor = 0$. Therefore only a finite number of terms in the above sum are non-zero. The number of non-zero terms in this sum is the largest integer k for which $p^k \leq n$.

By Legendre's formula the highest power of 2 that divides $16!$ is k, where

$$\begin{aligned} k &= \left\lfloor \frac{16}{2} \right\rfloor + \left\lfloor \frac{16}{2^2} \right\rfloor + \left\lfloor \frac{16}{2^3} \right\rfloor + \left\lfloor \frac{16}{2^4} \right\rfloor \\ &= \left\lfloor \frac{16}{2} \right\rfloor + \left\lfloor \frac{16}{4} \right\rfloor + \left\lfloor \frac{16}{8} \right\rfloor + \left\lfloor \frac{16}{16} \right\rfloor \\ &= \lfloor 8 \rfloor + \lfloor 4 \rfloor + \lfloor 2 \rfloor + \lfloor 1 \rfloor \\ &= 8 + 4 + 2 + 1 \\ &= 15. \end{aligned}$$

FOR INVESTIGATION

13.1 Why is 15 the only value of k for which $16! \div 2^k$ is an odd integer?

13.2 Use Legendre's formula to find the highest power of 3 that divides $16!$.

13.3 What is the highest power of 2 that divides $100!$?

13.4 What is the highest power of 3 that divides $100!$?

13.5 Find the prime factorization of $100!$

13.6 Find the least positive integer n such that 2^{100} is a factor of $n!$.

13.7 Prove that Legendre's formula is correct.

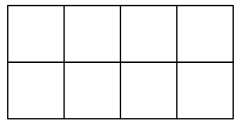
That is, show that the highest power of the prime p that divides $n!$ is given by the sum

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots + \left\lfloor \frac{n}{p^k} \right\rfloor,$$

where k is the largest integer such that $p^k \leq n$.

13.8 Let p be a prime number and let n be a positive integer. Find a formula in terms of p and n for the highest power of p that divides $(p^n)!$.

14. Diane has five identical blue disks, two identical red disks and one yellow disk. She wants to place them on the grid opposite so that each cell contains exactly one disk. The two red disks are not to be placed in cells that share a common edge.



How many different-looking completed grids can she produce?

A 96 B 108 C 144 D 180 E 216

Solution **B**

Note: The key to a problem of this kind is deciding the order in which to consider the placing of the differently coloured disks. In this case it is best to consider first the number of different ways the two red disks may be placed, because they are subject to the condition that they should not be put in cells that share an edge.

In the diagram we have labelled the cells so that we can refer to them.

P	Q	R	S
T	U	V	W

If the first red disk is placed the cell P, then the second red disk may be placed in any one of the 5 cells R, S, U, V and W.

Likewise, if it placed in the any of the cells S, T and Q, there are 5 possible cells for the second red disk.

If the first red disk is placed any one of the cells Q, R, U and V, there are 4 choices for the second red disk.

This gives $5 \times 4 + 4 \times 4 = 36$ ways to place the two red disks, but each possible pair of cells has been counted twice.

Therefore there are $36 \div 2 = 18$ different ways to place the two red disks.

Once the red disks have been placed, there remain 6 cells in which the yellow disk may be placed.

Once the red and yellow disks have been placed, the 5 blue disks must be placed in the remaining 5 empty cells. This may be done in just 1 way.

This gives $18 \times 6 \times 1 = 108$ different-looking ways of filling the grid.

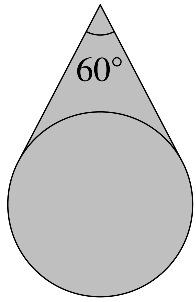
For investigation

14.1 How many different-looking grids can Diane produce if she has one yellow disk, three identical red disks and four identical blue disks, and there are no restrictions other than that each cell should contain exactly one disk?

14.2 How many different-looking grids can Diane produce if she has one yellow disk, two identical red disks, two identical blue disks and three identical green disks, and there are no restrictions other than that each cell should contain exactly one disk?

14.3 How many different-looking grids can Diane produce if she has one yellow disk, two identical red disks, two identical blue disks and three identical green disks, with each cell containing exactly one disk and the two red disks not in cells that share a common edge?

15. The shaded area shown in the diagram consists of the interior of a circle of radius 3 together with the area between the circle and two tangents to the circle. The angle between the tangents at the point where they meet is 60°.

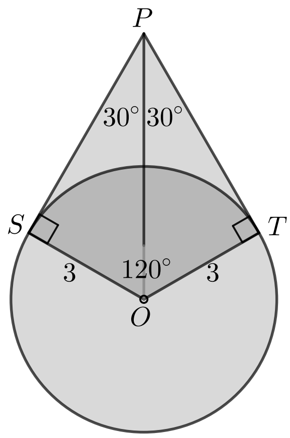


What is the shaded area?

A $6\pi + 9\sqrt{3}$ B $15\sqrt{3}$ C 9π
D $9\pi + 4\sqrt{3}$ E $6\pi + \frac{9\sqrt{3}}{4}$

Solution **A**

Let O be the centre of the circle and let PS and PT be the tangents to the circle, as shown.



In the triangles PSO and PTO we have $\angle PSO = \angle PTO = 90°$ because PS and PT are tangents to the circle; $SO = TO$, because they are radii of the same circle; and the side PO is common.

Therefore, the two triangles are congruent (RHS). Hence $\angle SPO = \angle TPO = 30°$.

It follows that $\dfrac{OS}{PS} = \tan 30° = \dfrac{1}{\sqrt{3}}$.

Hence $PS = OS\sqrt{3} = 3\sqrt{3}$.

Using the formula area $= \frac{1}{2}(\text{base} \times \text{height})$, for the area of a triangle, it follows that the area of the triangle PSO is $\frac{1}{2}(OS \times PS) = \frac{1}{2}(3 \times 3\sqrt{3}) = \frac{1}{2}(9\sqrt{3})$.

This is also the area of the congruent triangle PTO. Therefore the sum of the areas of these two triangles is $9\sqrt{3}$.

The total shaded area is the sum of the areas of these two triangle plus the area of that part of the circle that lies outside the two triangles. Because $\angle SOT = 120°$, the part of the circle outside the two triangles makes up two-thirds of the circle and hence its area is given by

$$\tfrac{2}{3}(\pi \times 3^2) = 6\pi.$$

Hence the shaded area is $6\pi + 9\sqrt{3}$.

For investigation

15.1 Explain why $\angle SOT = 120°$.

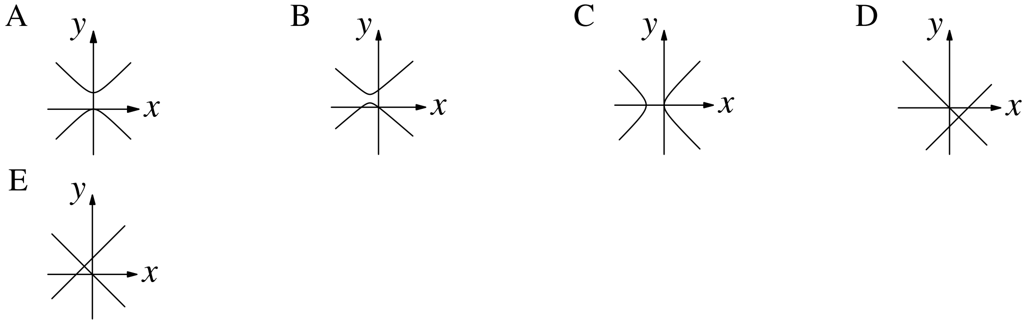
15.2 Explain why it follows that the area of the part of the circle outside the two triangles is two-thirds of the total area of the circle.

15.3 The solution above uses the fact that $\tan 30° = \dfrac{1}{\sqrt{3}}$.

Explain why this is correct.

15.4 What is the shaded area in the case where $\angle SPT = 120°$?

16. Which diagram represents the set of all points (x, y) satisfying $y^2 - 2y = x^2 + 2x$?



SOLUTION **E**

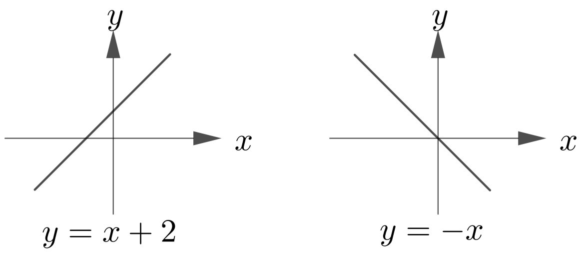
We have

$$\begin{aligned} y^2 - 2y = x^2 + 2x &\Leftrightarrow y^2 - x^2 - 2y - 2x = 0 \\ &\Leftrightarrow (y - x)(y + x) - 2(y + x) = 0 \\ &\Leftrightarrow (y - x - 2)(y + x) = 0 \\ &\Leftrightarrow y - x - 2 = 0 \quad \text{or} \quad y + x = 0 \\ &\Leftrightarrow y = x + 2 \quad \text{or} \quad y = -x. \end{aligned}$$

Note: An alternative method is to complete the square on both sides of the equation. See Problem 16.1 below.

We therefore see that the set of points satisfying the equation $y^2 - 2y = x^2 + 2x$ is made up of all the points on the line with the equation $y = x + 2$ together with all the points on the line with the equation $y = -x$.

The line with the equation $y = x + 2$ is the line with slope 1 that goes through the point $(0, 2)$.



The line with the equation $y = -x$ is the line with slope -1 that goes through the point $(0, 0)$.

The diagram that shows these two lines is that in option E.

FOR INVESTIGATION

16.1 Show that adding 1 to both sides of the equation $y^2 - 2y = x^2 + 2x$, provides an alternative way to show that $y = x + 2$ or $y = -x$.

16.2 What are the coordinates of the point where the lines with equations $y = x + 2$ and $y = -x$ meet?

16.3 Draw a diagram to represent the set of all points (x, y) which satisfy the equation $x^2 - 1 = y^2 + 2y$.

16.4 Draw a diagram to represent the set of all points (x, y) which satisfy the equation $x^2y^2 + xy = x^3 + y^3$.

17. The positive integers m, n and p satisfy the equation $3m + \dfrac{3}{n + \frac{1}{p}} = 17.$

What is the value of p?

A 2 B 3 C 4 D 6 E 9

Solution **A**

Because $\dfrac{3}{n + \frac{1}{p}} = 17 - 3m$, where m is an integer, it follows that $\dfrac{3}{n + \frac{1}{p}}$ is an integer.

Because n and p are positive integers, $1 < n + \frac{1}{p}$ and therefore $0 < \dfrac{3}{n + \frac{1}{p}} < 3$. Therefore $\dfrac{3}{n + \frac{1}{p}}$ is either 1 or 2.

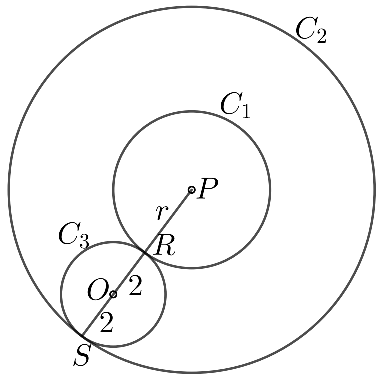
If $\dfrac{3}{n + \frac{1}{p}} = 1$, then $3m = 17 - \dfrac{3}{n + \frac{1}{p}} = 17 - 1 = 16$. This implies that $m = \frac{16}{3}$ contradicting the fact that m is an integer. Hence $\dfrac{3}{n + \frac{1}{p}} = 2$. Therefore $n + \frac{1}{p} = \frac{3}{2} = 1 + \frac{1}{2}$. Hence $n = 1$ and $p = 2$. Also $3m = 17 - \dfrac{3}{1 + \frac{1}{2}} = 17 - 2 = 15$ and so $m = 5$. We therefore see that $m = 5$, $n = 1$ and $p = 2$.

18. Two circles C_1 and C_2 have their centres at the point (3,4) and touch a third circle, C_3. The centre of C_3 is at the point (0,0) and its radius is 2.

What is the sum of the radii of the two circles C_1 and C_2?

A 6 B 7 C 8 D 9 E 10

Solution **E**



Let P be the centre of the circles C_1 and C_2, O be the centre of the circle C_3, and R and S be the points where C_1 and C_2, respectively, touch C_3. We let the radius of C_1 be r.

Note that the points O, P, R and S lie on a straight line.

By Pythagoras' Theorem, the distance between the point P with coordinates $(3, 4)$ and the point O with coordinates $(0, 0)$ is $\sqrt{3^2 + 4^2} = \sqrt{25} = 5$.

Therefore, $r + 2 = 5$ and hence $r = 3$.

The radius of the circle C_2 is $r + 2 + 2 = 3 + 2 + 2 = 7$.

Therefore the sum of the radii of C_1 and C_2 is $3 + 7 = 10$.

For investigation

18.1 Prove that the points P, R, O and S lie on a straight line.

19. The letters p, q, r, s and t represent different positive single-digit numbers such that $p - q = r$ and $r - s = t$.

How many different values could t have?

A 6 B 5 C 4 D 3 E 2

Solution **A**

Because $r = p - q$, we have $t = r - s = (p - q) - s = p - (q + s)$.

It follows that the maximum value of t occurs when p takes its maximum value, and $q + s$ takes its minimum value.

The maximum value of p is 9.

Because q and s have different values, the minimum value of $q + s$ is $1 + 2 = 3$.

Therefore the maximum possible value of t is $9 - 3 = 6$.

The following table shows that t can take all the positive integer values from 1 to 6.

p	q	r	s	t
9	1	8	2	6
9	1	8	3	5
9	2	7	3	4
9	2	7	4	3
9	3	6	4	2
9	3	6	5	1

Therefore the number of values that t could take is 6.

For investigation

19.1 In how many different ways is it possible to choose different single-digit values for p, q, r and s so that $t = 1$?

19.2 The letters p, q, r and s represent different positive single-digit numbers such that $p - q = r$ and $q - r = s$. How many different values could s have?

19.3 The letters p, q, r and s represent different positive single-digit numbers such that $p - q = r$ and $p - r = s$. How many different values could s have?

20. The real numbers x and y satisfy the equations $4^y = \dfrac{1}{8(\sqrt{2})^{x+2}}$ and $9^x \times 3^y = 3\sqrt{3}$.

What is the value of 5^{x+y}?

A $5\sqrt{5}$ B 5 C $\sqrt{5}$ D $\frac{1}{5}$ E $\frac{1}{\sqrt{5}}$

SOLUTION **E**

We have $4^y = (2^2)^y = 2^{2y}$, and $\dfrac{1}{8(\sqrt{2})^{x+2}} = \dfrac{1}{2^3(2^{\frac{1}{2}})^{x+2}} = 2^{-(3+\frac{1}{2}(x+2))}$.

Therefore from the equation $4^y = \dfrac{1}{8(\sqrt{2})^{x+2}}$ we deduce that $2y = -(3 + \frac{1}{2}(x+2))$.

Therefore

$$y = -\tfrac{1}{4}x - 2. \qquad (1)$$

Also, $9^x \times 3^y = (3^2)^x \times 3^y = 3^{2x+y}$ and $3\sqrt{3} = (3^1)(3^{\frac{1}{2}}) = 3^{1+\frac{1}{2}} = 3^{\frac{3}{2}}$.

Therefore, from the equation $9^x \times 3^y = 3\sqrt{3}$ we deduce that $2x + y = \frac{3}{2}$.

Therefore

$$y = -2x + \tfrac{3}{2}. \qquad (2)$$

From equations (1) and (2)

$$-\tfrac{1}{4}x - 2 = -2x + \tfrac{3}{2}.$$

This last equation may be rearranged to give

$$\tfrac{7}{4}x = \tfrac{7}{2}.$$

Hence $x = 2$.

Therefore, from equation (1), $y = -\frac{5}{2}$.

It follows that $x + y = 2 + (-\frac{5}{2}) = 2 - \frac{5}{2} = -\frac{1}{2}$.

Therefore $5^{x+y} = 5^{-\frac{1}{2}} = \frac{1}{\sqrt{5}}$.

FOR INVESTIGATION

20.1 The real numbers x and y satisfy the equations

$$(\sqrt{3})^x \times 3^y = \sqrt[5]{27}$$

and

$$8^x \times (\sqrt{2})^y = \sqrt[3]{4}.$$

Find the value of 8^{x+y}.

21. When written out in full, the number $(10^{2020} + 2020)^2$ has 4041 digits.

What is the sum of the digits of this 4041-digit number?

A 9 B 17 C 25 D 2048 E 4041

Solution **C**

Using the expansion $(x + y)^2 = x^2 + 2xy + y^2$, we have

$$\begin{aligned}(10^{2020} + 2020)^2 &= (10^{2020})^2 + 2 \times 10^{2020} \times 2020 + 2020^2\\ &= 10^{4040} + 4040 \times 10^{2020} + 4080400. \qquad (1)\end{aligned}$$

We now note that when we add the three terms 10^{4040}, 4040×10^{2002} and 408040, no two non-zero digits occur in the same column:

$$\begin{array}{rr} & 100\ldots\ldots\ldots\ldots0000\ldots\ldots\ldots\ldots0000000\\ + & 4040\ldots\ldots\ldots\ldots0000000\\ + & 4080400\\ \hline & 100\ldots\ldots\ldots\ldots4040\ldots\ldots\ldots\ldots4080400\end{array}$$

It follows that the non-zero digits in the final answer are exactly the non-zero digits in the three terms in (1) above.

Therefore the sum of the digits in the number $(10^{2020} + 2020)^2$ is

$$1 + 4 + 4 + 4 + 8 + 4 = 25.$$

Commentary

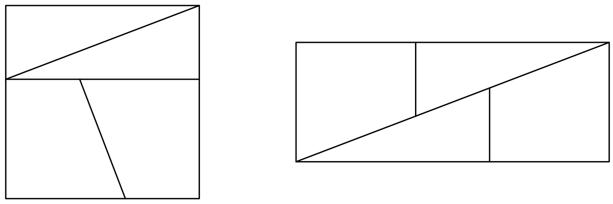
The exact number of 0s in the final answer is not important. What matters is that the non-zero digits in the three terms 10^{4040}, 4040×10^{2020} and 4080400 are in different columns, and so, when they are added, the non-zero digits in the final answer are exactly the non-zero digits in these terms. However, you are asked to work out the number of 0s in Problem 21.1 below.

For investigation

21.1 Find the positive integers m and n so that

$$(10^{2020} + 2020)^2 = 1\overbrace{000\ldots000}^{m}404\overbrace{000\ldots000}^{n}4080400.$$

22. A square with perimeter 4 cm can be cut into two congruent right-angled triangles and two congruent trapezia as shown in the first diagram in such a way that the four pieces can be rearranged to form the rectangle shown in the second diagram.

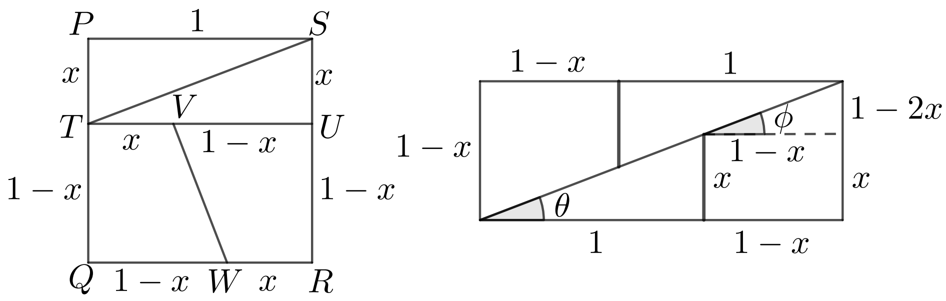


What is the perimeter, in centimetres, of this rectangle?

A $2\sqrt{5}$ B $4\sqrt{2}$ C 5 D $4\sqrt{3}$ E $3\sqrt{7}$

SOLUTION **A**

The square has perimeter 4 cm and hence its sides each have length 1 cm.



Let the vertices in the diagram on the left be labelled as shown.

We let the length of SU be x cm.

For the pieces to fit together as shown on the right, TV and WR must also have length x cm. It follows that the other lengths, in cm, are as shown.

METHOD 1

For the pieces to fit together the angles ϕ and θ must be equal, as shown on the right of the above diagram. Therefore $\tan\phi = \tan\theta$. That is,

$$\frac{1-2x}{1-x} = \frac{x}{1}.$$

Hence

$$1 - 2x = x(1 - x).$$

This last equation may be rearranged to give

$$x^2 - 3x + 1 = 0.$$

From the standard formula $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ for the solutions of a quadratic equation, we can deduce that

$$x = \frac{3 \pm \sqrt{5}}{2}.$$

Since $x < 1$ it follows that $x = \dfrac{3-\sqrt{5}}{2}$.

From the diagram on the right above we see that the perimeter of the rectangle is given by

$$(1-x)+(2-x)+(1-x)+(2-x) = 6-4x = 6-4\left(\frac{3+\sqrt{5}}{2}\right) = 2\sqrt{5}.$$

METHOD 2

Since they are made up from the same two triangles and two trapezia, the area of the rectangle equals the area of the square.

Therefore, from the diagram above, we have

$$(1+(1-x)) \times (1-x) = 1 \times 1,$$

that is,

$$(2-x)(1-x) = 1.$$

Hence

$$2-3x+x^2 = 1$$

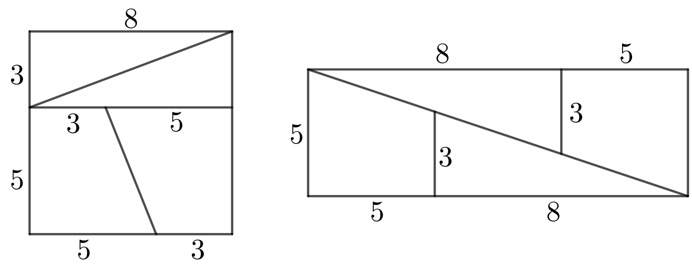
and therefore

$$x^2-3x+1 = 0.$$

Thus, as in Method 1, $x = \dfrac{3-\sqrt{5}}{2}$, and the perimeter of the rectangle is $2\sqrt{5}$.

FOR INVESTIGATION

22.1



The diagram above shows on the left an 8×8 square divided into four pieces which are rearranged to make the 13×5 rectangle shown on the right.

However $8 \times 8 < 13 \times 5$.

Where has the additional area come from?

Note: This puzzle has been attributed to William Hooper in his book *Rational Recreations* of 1774.

23. A function f satisfies $y^3 f(x) = x^3 f(y)$ and $f(3) \neq 0$. What is the value of $\dfrac{f(20) - f(2)}{f(3)}$?

A 6 B 20 C 216 D 296 E 2023

Solution **D**

By putting $x = 20$ and $y = 3$ in the equation $y^3 f(x) = x^3 f(y)$, we have $27f(20) = 8000f(3)$. Hence

$$f(20) = \frac{8000}{27} f(3).$$

By putting $x = 2$ and $y = 3$ in the same equation, we have $27f(2) = 8f(3)$. Hence

$$f(2) = \frac{8}{27} f(3).$$

Therefore,

$$\begin{aligned} f(20) - f(2) &= \frac{8000}{27} f(3) - \frac{8}{27} f(3) \\ &= \left(\frac{8000}{27} - \frac{8}{27}\right) f(3) \\ &= \frac{7992}{27} f(3) \\ &= 296 f(3). \end{aligned}$$

Therefore, because $f(3) \neq 0$, it follows that

$$\frac{f(20) - f(2)}{f(3)} = 296.$$

For investigation

23.1 What is the value of

$$\frac{f(46) - f(23)}{f(23)}?$$

23.2 Show that there are infinitely many positive integers x, y and z which satisfy the equation

$$\frac{f(x) - f(y)}{f(z)} = 7.$$

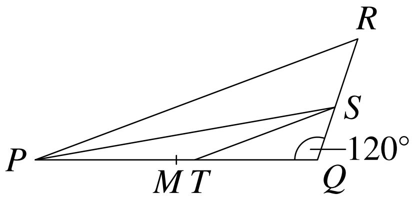
23.3 Show that for each positive integer n the equation

$$\frac{f(x) - f(y)}{f(z)} = n$$

has either infinitely many solutions in which x, y and z are positive integers, or no such solutions.

23.4 Show that a function g satisfies the equation $y^3 g(x) = x^3 g(y)$ for all real numbers x and y if, and only if, there is a constant k such that $g(x) = kx^3$, for all real numbers x.

24. In the diagram shown, M is the mid-point of PQ. The line PS bisects $\angle RPQ$ and intersects RQ at S. The line ST is parallel to PR and intersects PQ at T. The length of PQ is 12 and the length of MT is 1. The angle SQT is 120°.



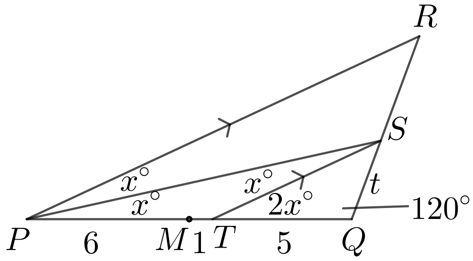
What is the length of SQ?

A 2 B 3 C 3.5 D 4 E 5

Solution **B**

Let t be the length of SQ.

Let $\angle QPS = x°$.



Because PQ has length 12 and M is the midpoint of PQ, it follows that MP has length 6. We are given that MT has length 1. Therefore PT has length 7, and TQ has length 5.

Because PS bisects $\angle RPQ$, we have $\angle RPS = \angle QPS = x°$. Therefore $\angle RPQ = 2x°$.

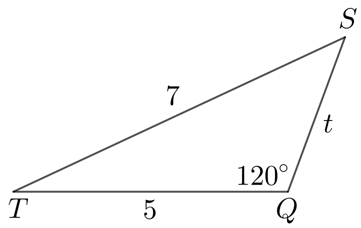
Because ST is parallel to PR, we have $\angle STQ = \angle RPQ = 2x°$.

It follows from the External Angle Theorem [See Problem 24.1], applied to the triangle PTS, that $\angle PST = x°$.

Therefore the triangle PST is isosceles and hence ST has the same length as PT, namely 7.

We now apply the Cosine Rule to the triangle STQ. This gives

$$7^2 = 5^2 + t^2 - 2(5t \cos 120°).$$



Hence, as $\cos 120° = -\frac{1}{2}$,

$$49 = 25 + t^2 + 5t.$$

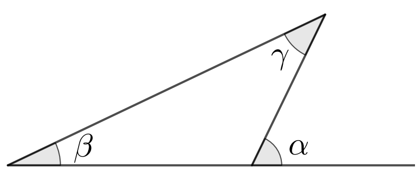
Therefore $t^2 + 5t - 24 = 0$. Hence $(t - 3)(t + 8) = 0$. Therefore t is either 3 or -8. Since t corresponds to a length, $t > 0$. We deduce that $t = 3$.

For investigation

24.1 The *External Angle Theorem* says that the external angle of a triangle is the sum of the two opposite angles.

In terms of the diagram it says that $\alpha = \beta + \gamma$.

Explain why the External Angle Theorem is true.

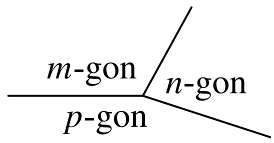


24.2 The solution above uses the fact that $\cos 120° = -\frac{1}{2}$. Explain why this is correct.

25. A regular m-gon, a regular n-gon and a regular p-gon share a vertex and pairwise share edges, as shown in the diagram.

What is the largest possible value of p?

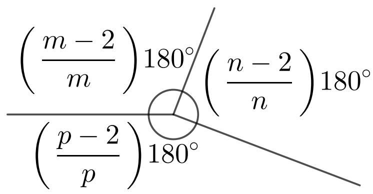
A 6 B 20 C 42 D 50 E 100



Solution **C**

The sum of the internal angles of a polygon with m sides is $(m-2)180°$ [see Problem 25.1]. Hence each internal angle of the regular m-gon is $\left(\frac{m-2}{m}\right)180°$, and similarly for the regular n-gon and the regular p-gon. Therefore, because the sum of the angles at a point is 360°, we have

$$\left(\frac{m-2}{m}\right)180 + \left(\frac{n-2}{n}\right)180 + \left(\frac{p-2}{p}\right)180 = 360.$$



It follows that

$$\frac{m-2}{m} + \frac{n-2}{n} + \frac{p-2}{p} = 2.$$

We may rewrite this last equation as $\left(1-\frac{2}{m}\right) + \left(1-\frac{2}{n}\right) + \left(1-\frac{2}{p}\right) = 2$. Hence

$$\frac{1}{m} + \frac{1}{n} + \frac{1}{p} = \frac{1}{2}. \qquad (1)$$

We note that as p is a positive integer, it follows from (1) that $\frac{1}{m} + \frac{1}{n} < \frac{1}{2}$. (2)

We seek a solution of (1), where m, n and p are positive integers and p is as large as possible.

For p to be as large as possible, $\frac{1}{p}$ needs to be as small as possible. Hence, by (1), $\frac{1}{m} + \frac{1}{n}$ needs to be as large as possible, subject to the inequality (2). Thus m and n need to be as small as possible. Without loss of generality we may assume that $m \leq n$.

By (2), $m \geq 3$. If $m = 3$, then $\frac{1}{n} < \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, and hence $n > 6$. When $m = 3$ and $n = 7$, we have $\frac{1}{m} + \frac{1}{n} = \frac{1}{3} + \frac{1}{7} = \frac{10}{21}$. Hence, by (1), $\frac{1}{p} = \frac{1}{2} - \frac{10}{21} = \frac{1}{42}$ and so $p = 42$. We show that this gives the largest possible value of p.

We cannot have $m = n = 4$, as this is not compatible with the inequality (2).

If $n > m \geq 4$, we have $\frac{1}{m} + \frac{1}{n} \leq \frac{1}{4} + \frac{1}{5} = \frac{9}{20} < \frac{10}{21}$ and so $\frac{1}{m} + \frac{1}{n}$ is not as large as possible.

Hence the largest possible value of p is given by $m = 3$, $n = 7$ and $p = 42$.

For investigation

25.1 Prove that the sum of the angles of a polygon with m sides is $(m-2)180°$ and hence that each internal angle of a regular m-gon is $\left(\frac{m-2}{m}\right)180°$.