Senior Mathematical Challenge

Thursday 6 November 2014

Organised by the United Kingdom Mathematics Trust

supported by Institute and Faculty of Actuaries

Solutions and investigations

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some exercises for further investigation. We welcome comments on these solutions. Please send them to enquiry@ukmt.org.uk.

The Senior Mathematical Challenge (SMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the SMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. So for each question we have included a complete solution which does not use the fact that one of the given alternatives is correct. Thus we have aimed to give full solutions with all steps explained. We therefore hope that these solutions can be used as a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the British Mathematical Olympiad and similar competitions).

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
C	B	D	A	C	B	B	C	D	C	E	D	E	E	C	A	C	D	E	A	B	A	E	A	B

1. What is 98×102?

A 200 B 9016 C 9996 D 998 E 99 996

SOLUTION

C There are, of course, several ways to evaluate the product. In the absence of a calculator the method which involves least effort is to exploit the *difference of two squares* identity, $(x - y)(x + y) = x^2 - y^2$. This gives

$$98 \times 102 = (100 - 2)(100 + 2) = 100^2 - 2^2 = 10\,000 - 4 = 9996.$$

Note that there are two useful checks that could be used here.

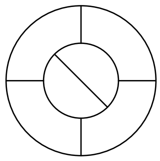
(a) Because $8 \times 2 = 16$, the units digit of 98×102 is 6. This rules out options A and D.

(b) Because 98 and 102 are each close to 100, their product is close to $100 \times 100 = 10\,000$. This rules out all the options other than C.

FOR INVESTIGATION

1.1 What is 998×1002?

2. The diagram shows 6 regions. Each of the regions is to be painted a single colour, so that no two regions sharing an edge have the same colour.



What is the smallest number of colours required?

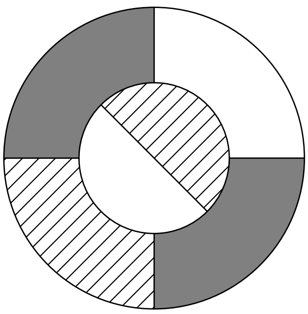
A 2 B 3 C 4 D 5 E 6

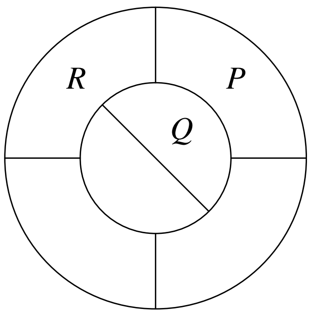
SOLUTION

B The figure on the left below shows that with just three colours we can paint the six regions so that regions sharing an edge are painted different colours. This shows that at most three colours are needed.

Since each pair of the regions labelled P, Q and R in the figure on the right shares an edge, these three regions must be painted different colours. So at least three colours are needed.

Therefore three is the smallest number of colours required.





3. December 31st 1997 was a Wednesday.

How many Wednesdays were there in 1997?

A 12 B 51 C 52 D 53 E 365

Solution

D As 1997 was not a leap year, there were 365 days in 1997. Since $365 = 52 \times 7 + 1$, the year 1997 was made up of 52 periods of 7 days, together with December 31st which was a Wednesday. So each of the 52 preceding periods of 7 days began with a Wednesday. Therefore there were 53 Wednesdays in 1997.

4. After I had spent $\frac{1}{5}$ of my money and then spent $\frac{1}{4}$ of what was left, I had £15 remaining.

How much did I start with?

A £25 B £75 C £100 D £135 E £300

Solution

A Suppose that I started with £x. After spending $\frac{1}{5}$ of my money, I was left with $\frac{4}{5}$ of what I started with. After spending $\frac{1}{4}$ of what was left, there remained $\frac{3}{4} \times \frac{4}{5} = \frac{3}{5}$ of what I started with. Since £15 remained, $\frac{3}{5}x = 15$. Therefore $x = \frac{5}{3} \times 15 = 25$. Therefore, I started with £25.

5. How many integers between 1 and 2014 are multiples of both 20 and 14?

A 7 B 10 C 14 D 20 E 28

Solution

C An integer is a multiple of both 20 and 14 if, and only if, it is a multiple of their least common multiple. The least common multiple of 20 and 14 is 140. The integers between 1 and 2014 that are multiples of 140 are 140, 280, 420, 560, ... and so on. So the number of integers between 1 and 2014 that are multiples of 140 and hence multiples of both 14 and 20 is the integer part of $\frac{2014}{140}$. Now

$$\frac{2014}{140} = 14 + \frac{54}{140},$$

and it follows that there are 14 integers between 1 and 2014 that are multiples of both 20 and 14.

For investigation

5.1 How many integers are there between 1 and 1000 that are multiples of both 6 and 21?

5.2 How many integers are there between 1 and 10^6 that are multiples of 2, 3, 5 and 7?

5.3 Show that for all positive integers a and b, an integer is a multiple of both a and b if, and only if, it is a multiple of the least common multiple of a and b.

6. In the addition sum shown, each of the letters T, H, I and S represents a non-zero digit.

$$\begin{array}{r} T\,H\,I\,S \\ +\quad I\,S \\ \hline 2\,0\,1\,4 \end{array}$$

What is $T + H + I + S$?

A 34 B 22 C 15 D 9 E 7

Solution

B The sum of '$THIS$' and 'IS' has 4 in the units place. Hence S is either 2 or 7. The digit 1 in the tens column of the sum is odd. If this digit just came from $I + I$, it would be even. So there is a *carry* from the units column to the tens column. Therefore S is 7.

Because I is non-zero, it follows that I is 5. Hence there is a *carry* from the tens column to the hundreds column. Therefore H is 9, and so T is 1.

Therefore $T + H + I + S = 1 + 9 + 5 + 7 = 22$.

For investigation

6.1 Are there any other solutions if the restriction that T, H, I and S are non-zero is dropped?

7. According to recent research, global sea levels could rise 36.8 cm by the year 2100 as a result of melting ice.

Roughly how many millimetres is that per year?

A 10 B 4 C 1 D 0.4 E 0.1

Solution

B The year 2100 is $2100 - 2015 = 85$ years away, and 36.8 cm is 368 millimetres. Now $\frac{368}{85}$ is approximately $\frac{360}{90}$, that is, 4.

Remark

The key idea here is to replace the numbers 368 and 85 by approximations, say, a and b, so that the fraction $\frac{a}{b}$ is easy to evaluate and is sufficiently close to $\frac{368}{85}$. Other choices are possible.

For investigation

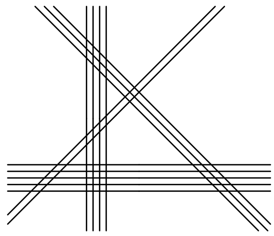
7.1 Find *without using a calculator* approximations to the values of the following fractions (a) $\frac{4783}{77}$, (b) $\frac{34\,689}{683}$, (c) $\frac{725\,546}{239}$.

7.2 The mean distance of the earth from the sun is 1.496×10^8 km. The velocity of light is 299 792 km per second. *Without using a calculator* estimate the number of minutes it takes light from the sun to reach the earth.

8. The diagram shows four sets of parallel lines, containing 2, 3, 4 and 5 lines respectively.

How many points of intersection are there?

A 54 B 63 C 71 D 95 E 196



Solution

C When a set of p parallel lines intersects a set of q parallel lines, each line of the first set meets each line of the second set and so there are $p \times q$ points of intersection. Here there are 4 sets of parallel lines, and there are 6 pairs of these sets that intersect each other. The total number of intersections is $2\times3+2\times4+2\times5+3\times4+3\times5+4\times5 = 6+8+10+12+15+20 = 71$.

9. Which of the following is divisible by 9?

A $10^{2014}+5$ B $10^{2014}+6$ C $10^{2014}+7$ D $10^{2014}+8$ E $10^{2014}+9$

Solution

D When written in standard notation, the number $10^{2014}-1$ consists of a string of 2014 consecutive occurrences of the digit 9. It follows that $10^{2014}-1$ is divisible by 9.

Another way to see this is to use the identity $x^n-1=(x-1)(x^{n-1}+x^{n-2}+\cdots+x+1)$. It follows that x^n-1 is divisible by $x-1$. Hence, putting $x=10$ and $n=2014$, we see that $10^{2014}-1$ is divisible by 9. For a third method see Problem 9.3.

Therefore when 10^{2014} is divided by 9, the remainder is 1. That is, $10^{2014}=9n+1$ for some integer n.

It follows that $10^{2014}+5=9n+6$, $10^{2014}+6=9n+7$, $10^{2014}+7=9n+8$, $10^{2014}+8=9n+9=9(n+1)$ and $10^{2014}+9=9n+10=9(n+1)+1$. From this we see that $10^{2014}+8$ is divisible by 9, but no other number given as an option is divisible by 9.

For investigation

9.1 For which value of n, with $1 \le n \le 9$, is $10^{2015}+n$ divisible by 9?

9.2 What is the remainder when 10^{2014} is divided by 11?

9.3 Another method is to use the fact that:

When a positive integer is divided by 9 the remainder is the same as when the sum of its digits is divided by 9.

Note that it follows that:

A positive integer is divisible by 9 if, and only if, the sum of its digits is divisible by 9.

(a) Use these facts to answer Question 9.

(b) Explain why the facts above are correct.

10. A rectangle has area $120\,\text{cm}^2$ and perimeter 46 cm.

Which of the following is the length of each of the diagonals?

A 15 cm B 16 cm C 17 cm D 18 cm E 19 cm

SOLUTION

C Suppose that the length of the rectangle is a cm and its width is b cm. The rectangle has area $120\,\text{cm}^2$, and therefore $ab = 120$. The rectangle has perimeter 46 cm, and therefore $2a + 2b = 46$. Hence $a + b = 23$. By Pythagoras' Theorem, the length of the diagonal, in centimetres, is $\sqrt{a^2 + b^2}$. Now

$$a^2 + b^2 = (a + b)^2 - 2ab = 23^2 - 240 = 529 - 240 = 289 = 17^2.$$

It follows that $\sqrt{a^2 + b^2} = 17$, and hence that each diagonal has length 17 cm.

REMARKS

Notice that in the above solution we did not need to work out the values of a and b separately in order to evaluate $a^2 + b^2$ because we are able to express $a^2 + b^2$ in terms of $a + b$ and ab.

We can generalize this. A polynomial in more than one unknown is said to be a *symmetric polynomial* if, however, we swap round the unknowns, the resulting polynomial is equivalent to the one we started with. For example, the polynomial $a + b$ is symmetric because if we swap round a and b we get the polynomial $b + a$, which is equivalent to the polynomial $a + b$ that we started with.

Another example is the polynomial $a^2b + ab^2$. If we swap round a and b in this polynomial we obtain the polynomial $b^2a + ba^2$, which is equivalent to the polynomial we started with.

We can have symmetric polynomials with more than two unknowns. For example, the polynomial $ab + bc + ca$ is a symmetric polynomial in the three unknowns a, b and c. If, for example, in this polynomial we replace a by c, c by b and b by a, we obtain the polynomial $ca + ab + bc$ which is equivalent to the original polynomial.

The *elementary symmetric* polynomials in the two unknowns a and b are the polynomials $a + b$ and ab. The *Fundamental Theorem on Symmetric Polynomials* says that every other symmetric polynomial in the two unknowns a and b and with integer coefficients can be obtained from these elementary symmetric polynomials just using addition, subtraction and multiplication.

For example, in the solution to Question 10 we obtained $a^2 + b^2$ by multiplying the elementary symmetric polynomial $a + b$ by itself to get $(a + b)^2$, and then subtracting the elementary symmetric polynomial ab twice, so that we ended up with $(a + b)^2 - 2ab$.

The *elementary symmetric polynomials* in the three unknowns a, b and c are $a+b+c$, $ab+bc+ca$ and abc. The *Fundamental Theorem* also tells us that every symmetric polynomial in the three unknowns a, b and c with integer coefficients can be obtained from the elementary ones using just addition, subtraction and multiplication. It can be generalized to cover the case of symmetric polynomials in any finite number of unknowns.

You are asked to explore these ideas in the following problems, after first being asked to find the values of a and b in Question 10.

For investigation

10.1 Find the length and width of the rectangle in Question 10.

10.2 Which of the following are symmetric polynomials?

(a) $a^5 + b^5$,

(b) $2a^2b + 3ab^2$,

(c) $a^2b + b^2c + c^2a$,

(d) $a^4 + b^4 + c^4$.

10.3 Express the following symmetric polynomials in terms of the elementary symmetric polynomials $a + b$ and ab.

(a) $a^2b + ab^2$,

(b) $a^3 + b^3$,

(c) $a^4 + b^4$.

10.4 You are given that $a + b = 7$ and $ab = 5$. Evaluate $a^3 + b^3$.

10.5 Express the symmetric polynomial $a^3 + b^3 + c^3$ in terms of the elementary symmetric polynomials $a + b + c$, $ab + bc + ca$ and abc.

10.6 Find a proof of the *Fundamental Theorem on Symmetric Polynomials*.

11. A Mersenne prime is a prime of the form $2^p - 1$, where p is also a prime.

One of the following is *not* a Mersenne prime. Which one is it?

A $2^2 - 1$ B $2^3 - 1$ C $2^5 - 1$ D $2^7 - 1$ E $2^{11} - 1$

Solution

E The options have the form $2^p - 1$, for $p = 2, 3, 5, 7$ and 11, respectively. All these values of p are primes. So we need to find out which of the options is not itself a prime.

We see that $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 31$ and $2^7 - 1 = 127$ are all prime. So it must be that the remaining option, $2^{11} - 1$, is not prime.

In the context of the SMC that is all you need do, but for a complete solution we need to check that $2^{11} - 1$ is indeed not prime. This is not immediately obvious, but you can check that $2^{11} - 1 = 2047 = 23 \times 89$ and so is not prime.

For investigation

11.1 Show that if p is a positive integer which is not prime, then $2^p - 1$ is not a prime number.

Hint: Let $p = ab$, where a and b are integers each greater than 1. Then $2^p = 2^{ab} - 1 = (2^a)^b = x^b - 1$, where $x = 2^a$.

Note that this shows that for $2^p - 1$ to be prime, it is necessary for p to be a prime number. The fact that $2^{11} - 1$ is not prime shows that this is not a sufficient condition.

11.2 One reason for being interested in Mersenne primes is that if $2^p - 1$ is a Mersenne prime, then the number $2^{p-1}(2^p - 1)$ is a perfect number. A positive integer n is said to be a *perfect number* if it is equal to the sum of all its divisors, including 1, but not n itself. For example, as $2^2 - 1$ is a Mersenne prime, the number $2^{2-1}(2^2 - 1) = 2 \times 3 = 6$ is perfect. This is easy to check. The divisors of 6, other than 6 itself, are 1, 2 and 3, and $1 + 2 + 3 = 6$.

$2^3 - 1$ and $2^5 - 1$ are both Mersenne primes. Check by direct calculations that both $2^{3-1}(2^3 - 1)$ and $2^{5-1}(2^5 - 1)$, that is, 28 and 496, are perfect numbers.

11.3 Show that if $2^p - 1$ is a prime number, then $2^{p-1}(2^p - 1)$ is a perfect number.

11.4 Show that if n is an even number that is perfect, then there is a Mersenne prime $2^p - 1$ such that $n = 2^{p-1}(2^p - 1)$.

Hint: First, it is useful to introduce the standard notation $\sigma(n)$ for the sum of all the divisors of n, including n itself. Since this sum includes the divisor n, a positive integer, n is perfect if, and only if, $\sigma(n) = 2n$.

Second, use the fact that each even positive integer can be written in the form $2^a(2b - 1)$, where a and b are positive integers.

(a) Now show that $\sigma(2^a(2b - 1)) = (1 + 2 + 2^2 + \cdots + 2^a)\sigma(2b - 1)$
$= (2^{a+1} - 1)\sigma(2b - 1)$.

(b) Deduce that $2^a(2b-1)$ is perfect if, and only if, $(2^{a+1} - 1)\sigma(2b - 1) = 2^{a+1}(2b-1)$.

(c) Deduce that $b = 2^a$ and $2^{a+1} - 1$ is prime.

Remarks

The proof that if $2^p - 1$ is prime, then $2^{p-1}(2^p - 1)$ is a perfect number is to be found in Euclid's *Elements* (Book 9, Proposition 36).

Marin Mersenne (1588–1648) was a French monk. At a time long before research in mathematics was published in journals, and before the internet, he played an important role by corresponding with mathematicians and communicating their results. He tried to find a general formula for prime numbers, and gave a list of prime numbers p for which he thought $2^p - 1$ was a prime, but he made a number of errors.

At the time of writing (October 2014), there are 48 known Mersenne primes of which the largest is $2^{57\,865\,161} - 1$. So there are 48 known even perfect numbers of which the largest is $2^{57\,865\,160}(2^{57\,865\,161} - 1)$. It is not known whether there are infinitely many Mersenne primes, nor is it known whether there are any odd perfect numbers.

12. Karen has three times the number of cherries that Lionel has, and twice the number of cherries that Michael has. Michael has seven more cherries than Lionel.

How many cherries do Karen, Lionel and Michael have altogether?

A 12 B 42 C 60 D 77 E 84

Solution

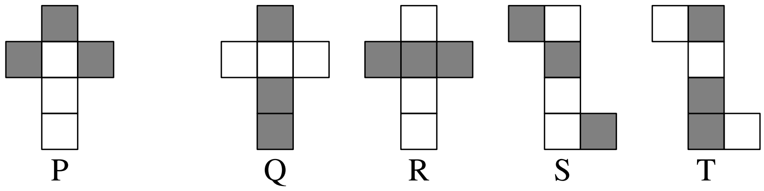
D Suppose that Karen has x cherries. Then Lionel has $\frac{1}{3}x$ cherries and Michael has $\frac{1}{2}x$ cherries. Michael has seven more cherries than Lionel and so $\frac{1}{2}x - \frac{1}{3}x = 7$. Therefore $(\frac{1}{2} - \frac{1}{3})x = 7$, that is, $(\frac{3-2}{6})x = 7$, and hence $\frac{1}{6}x = 7$. Therefore $x = 42$. It follows that Karen has 42 cherries, Lionel has 14 cherries and Michael has 21 cherries. So they have $42 + 14 + 21 = 77$ cherries between them.

For investigation

12.1 The fractions in this solution could be avoided by supposing that Karen has $6x$ cherries. Rework the solution using this assumption.

12.2 Find a formula for the total number of cherries they have between them if Karen has p times the number of cherries that Lionel has, and q times the number of cherries that Michael has, and Michael has c more cherries than Lionel.

13. Each of the five nets P, Q, R, S and T is made from six squares. Both sides of each square have the same colour. Net P is folded to form a cube.



How many of the nets Q, R, S and T can be folded to produce a cube that looks the same as that produced by P?

A 0 B 1 C 2 D 3 E 4

Solution

E Each net folds to make a cube with three white and three grey faces. There are two possibilities for such a cube. Either the three grey faces meet at a common vertex, or they do not have a vertex in common.

If the three grey faces meet at a common vertex, the cube looks like the one on the left below, where the top, back and right-hand faces are grey.

If the grey faces do not have a vertex in common then the cube looks like the one on the right below, where the top, left-hand and right-hand faces are grey.

It the nets are folded, the nets P, Q, R, S and T make cubes where the three grey faces do not have a vertex in common. So they all form a cube which looks like the cube on the right above. Hence all four of Q, R, S and T fold to make a cube that looks the same as that produced by P.

14. Given that $\dfrac{3x+y}{x-3y}=-1$, what is the value of $\dfrac{x+3y}{3x-y}$?

A -1 B 2 C 4 D 5 E 7

Solution

E From $\dfrac{3x+y}{x-3y}=-1$, it follows that $3x+y=-(x-3y)$, that is, $3x+y=-x+3y$. Hence $4x=2y$ and so $y=2x$. Therefore

$$\begin{aligned}\frac{x+3y}{3x-y}&=\frac{x+6x}{3x-2x}\\&=\frac{7x}{x}\\&=7.\end{aligned}$$

For investigation

14.1 It is, in fact, possible to evaluate $\dfrac{x+3y}{3x-y}$ directly from the given value for $\dfrac{3x+y}{x-3y}$ without the need to find y in terms of x. However this involves rather more work!

(a) Show that

$$\frac{x+3y}{3x-y}=\frac{3\left(\dfrac{3x+y}{x-3y}\right)-4}{4\left(\dfrac{3x+y}{x-3y}\right)+3}.$$

(b) Use the result of part (a) to verify that $\dfrac{x+3y}{3x-y}=7$.

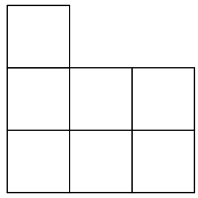
14.2 (a) Find values for a, b, c and d so that

$$\frac{a\left(\dfrac{3x+y}{x-3y}\right)+b}{c\left(\dfrac{3x+y}{x-3y}\right)+d}=\frac{x+y}{x-y}.$$

(b) Use the result of part (a) to evaluate $\dfrac{x+y}{x-y}$, given that $\dfrac{3x+y}{x-3y}=5$.

15. The figure shown alongside is made from seven small squares. Some of these squares are to be shaded so that:

(i) at least two squares are shaded;
(ii) two squares meeting along an edge or at a corner are not both shaded.

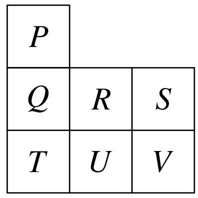


How many ways are there to do this?

A 4 B 8 C 10 D 14 E 18

SOLUTION

C We solve this problem by listing all the shadings which satisfy the given conditions. It helps to label the squares as shown. We can then indicate a particular shading by specifying the squares that have been shaded. For example, by "the shading PST" we mean the shading in which the squares labelled P, S and T are those that are shaded.



We say that a shading is *correct* if it satisfies the conditions (i) and (ii) specified in the question.

To ensure that we get the right answer we need to list the correct shadings in a systematic way so that we can be sure that only correct shadings are included in our list, and that every correct shading occurs exactly once.

We do this by considering the squares in the alphabetical order of their labels.

So we first consider correct shadings in which P is shaded. We see that if P is shaded then Q and R cannot be shaded. If, in addition to P, we shade S, then the only other square that could be shaded is T. So there are just two correct shadings in which P and S are shaded, namely PS and PST.

If P is shaded but S is not shaded, then the only possibilities are to to shade just T, or T and V, or just U, or just V. So there are four correct shadings PT, PTV, PU and PV in which P is shaded but S is not shaded.

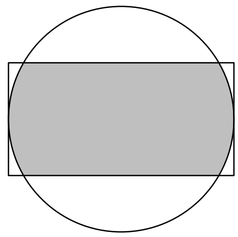
Next we consider the cases where P is not shaded. If then Q is shaded, we cannot shade R, Tor U and the only possibility is to shade just one of S and V. So we get two more correct shadings QS and QV.

If we continue in this way we obtain the following list of all the correct shadings: PS, PST, PT, PTV, PU, PV, QS, QV, ST, TV. We see that there are 10 of them.

16. The diagram shows a rectangle measuring 6×12 and a circle.

The two shorter sides of the rectangle are tangents to the circle. The circle and rectangle have the same centre.

The region that lies inside both the rectangle and the circle is shaded. What is its area?

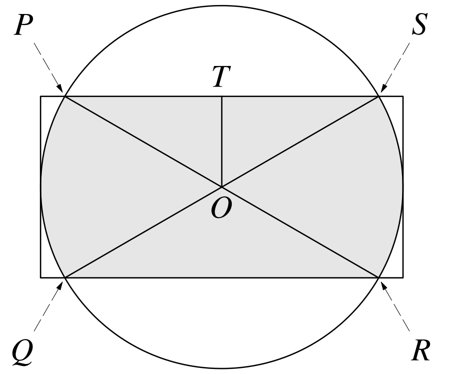


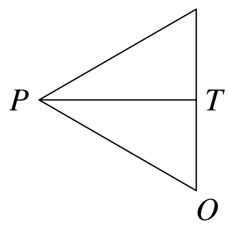
A $12\pi + 18\sqrt{3}$ B $24\pi - 3\sqrt{3}$ C $18\pi - 8\sqrt{3}$
D $18\pi + 12\sqrt{3}$ E $24\pi + 18\sqrt{3}$

SOLUTION

A We let O be the centre of the circle and we let P, Q,R and S be the points where the rectangle meets the circle, as shown.

The shaded region is made up of the two triangles, POS and QOR, and the two sectors OPQ and ORS of the circle. We calculate their areas separately.





We first note that, as the shorter sides of the rectangle are tangents to the circle, the radius of the circle is half the length of the rectangle. So the circle has radius 6.

Let T be the point where the perpendicular from O to PS meets PS. In the right-angled triangle POT, the hypotenuse OP has length 6, as it is a radius of the circle, and OT, being half the width of the rectangle, has length 3. Therefore by Pythagoras' Theorem the length of PT is $\sqrt{6^2 - 3^2} = \sqrt{27} = 3\sqrt{3}$. Similarly $TS = 3\sqrt{3}$. Therefore PS has length $6\sqrt{3}$ and the area of triangle OPS is $\frac{1}{2}(\text{base} \times \text{height}) = \frac{1}{2}(6\sqrt{3} \times 3) = 9\sqrt{3}$. Similarly, triangle QOR has area $9\sqrt{3}$.

Since OT is half the length of OP, the triangle OTP is half of an equilateral triangle. It follows that $\angle POT = 60°$. Similarly $\angle SOT = 60°$. Because the angles at O on the straight line QOS have sum 180°, it follows that, also, $\angle POQ = 60°$.

Therefore the area of the sector OPQ is one-sixth that of the circle. The circle has radius 6, and hence its area is $\pi(6^2)$, that is, 36π. So the area of the sector OPQ is $\frac{1}{6}(36\pi) = 6\pi$. Similarly, the sector ORS has area 6π.

Therefore the area that is shaded is $(2 \times 9\sqrt{3}) + (2 \times 6\pi) = 18\sqrt{3} + 12\pi$.

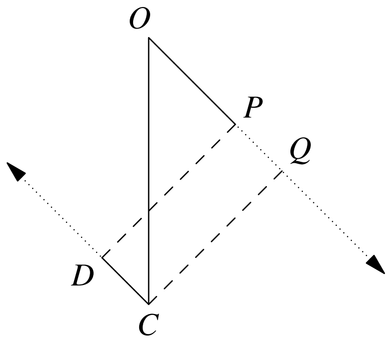
17. An oil tanker is 100 km due north of a cruise liner. The tanker sails SE at a speed of 20 kilometres per hour and the liner sails NW at a speed of 10 kilometres per hour.

What is the shortest distance between the two boats during the subsequent motion?

A 100 km B 80 km C $50\sqrt{2}$ km D 60 km E $33\frac{1}{3}$ km

Solution

C Let O and C be the initial positions of the oil tanker and the cruise liner, and let P and D be their positions when they are their shortest distance apart.



Because the oil tanker is sailing SE, $\angle COP = 45°$. Because the cruise liner is sailing NW, $\angle DCO = 45°$.

Because the alternate angles, $\angle COP$ and $\angle DCO$, are equal, CD is parallel to OP. Therefore PD is perpendicular to the paths of both ships, and hence has the same length as the perpendicular, CQ, from C to the path of the oil tanker. Let this length be x km.

In the triangle OQC, $\angle OQC$ is a right angle and $\angle COQ = 45°$. It follows that the triangle OQC is a right-angled isosceles triangle. So the length of OQ is the same as that of CQ, namely x km. The tanker is initially 100 km north of the liner and therefore OC has length 100 km.

By Pythagoras' Theorem applied to this triangle, $x^2 + x^2 = 100^2$. So $2x^2 = 100^2$. It follows that

$$x = \frac{100}{\sqrt{2}} = 50\sqrt{2}.$$

Alternatively, we could use trigonometry, to give $x = 100\cos 45° = 100 \times \frac{1}{\sqrt{2}} = 50\sqrt{2}$.

Remark

Note that in order to solve this problem we did not need to locate the positions of the ships when they are at their shortest distance apart. The speeds of the ships are not relevant. These speeds affect the time and the positions of the ships when they are at their closest distance apart, but not what this distance is.

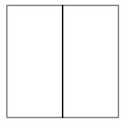
For investigation

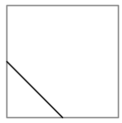
17.1 Assume that the two ships set off at the same time. How far will each of them have travelled when they are at their shortest distance apart?

17.2 Assume that the two ships set off at the same time. How long does it take them to reach the positions at which they are at their shortest distance apart?

17.3 The question and the solution both implicitly assume that the ships are on a flat sea to which the standard facts of two-dimensional Euclidean geometry are applicable, rather than on the almost spherical surface of the Earth. How good an approximation to the actual shortest distance do you think this gives?

18. Beatrix decorates the faces of a cube, whose edges have length 2. For each face, she either leaves it blank, or draws a single straight line on it. Every line joins the midpoints of two edges, either opposite or adjacent, as shown.



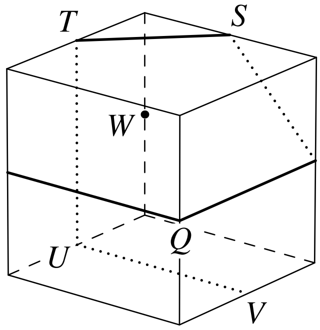


What is the length of the longest unbroken line that Beatrix can draw on the cube?

A 8 B $4 + 4\sqrt{2}$ C $6 + 3\sqrt{2}$ D $8 + 2\sqrt{2}$ E 12

Solution

D Since a cube has 6 faces, no unbroken path can contain more than 6 lines. The diagram shows an unbroken path $PQRSTUV$ made up of 6 lines: 4 lines joining midpoints of opposite edges, each of length 2; and 2 lines joining midpoints of adjacent edges, each of length $\sqrt{2}$. So the path shown has length $4 \times 2 + 2 \times \sqrt{2} = 8 + 2\sqrt{2}$.



To complete the solution, we need to show that it is not possible to draw a longer unbroken line on the cube.

It will be convenient to call a line joining the midpoint of *opposite* edges an o-line, and a line joining the midpoints of *adjacent* edges an a-line.

We cannot have a path made up of 6 o-lines, because 4 consecutive o-lines form a closed path which cannot be extended.

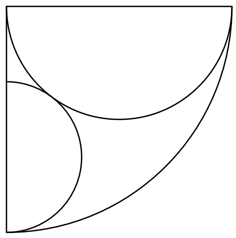
So to have a path whose length is greater than $8 + 2\sqrt{2}$ (which consists of 4 o-lines and 2 a-lines) we would need to have 5 o-lines and 1 a-line. A path of this form, if it exists, as it includes just one a-line, must include at least three consecutive o-lines, and, by the earlier remark, it cannot include 4 consecutive o-lines.

Now consider a path made up of 3 consecutive o-lines. For example, consider the path $PQRW$. As we have seen, if we extend this path with a fourth o-line it becomes closed and we could not continue it any further. So to obtain a path with 5 o-lines and 1 a-line we would need to continue it with an a-line, for example WT, to make the path $PQRWT$. But now the only line that could be added to this path is either an a-line or an o-line on the top face. We could not add a sixth line to this path. This would also be true if we extended the path $PQRW$ by adding the a-line UP.

So there is no path made up of 5 o-lines and 1 a-line.

Hence the longest path that can be drawn has length $8 + 2\sqrt{2}$.

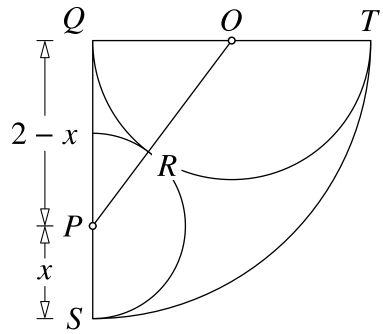
19. The diagram shows a quadrant of radius 2, and two touching semicircles. The larger semicircle has radius 1. What is the radius of the smaller semicircle?



A $\frac{\pi}{6}$ B $\frac{\sqrt{3}}{2}$ C $\frac{1}{2}$ D $\frac{1}{\sqrt{3}}$ E $\frac{2}{3}$

SOLUTION

E Let the smaller semicircle have radius x. Let O be the centre of the larger semicircle and let P be the centre of the smaller semicircle. Let Q be the centre of the quadrant, let R be the point where the semicircles meet, and let S and T be the points shown in the diagram.



Because OQ is a radius of the larger semicircle, it has length 1. As PS is a radius of the smaller circle it has length x, and so PQ has length $2 - x$. Since the two semicircles touch at R, the points O, R and P are in a straight line and hence OP has length $1 + x$.

Therefore, by Pythagoras' Theorem applied to the right-angled triangle OQP,

$$(1 + x)^2 = 1^2 + (2 - x)^2.$$

This equation can be expanded to give

$$1 + 2x + x^2 = 1 + (4 - 4x + x^2).$$

It follows that

$$6x = 4,$$

and therefore

$$x = \frac{2}{3}.$$

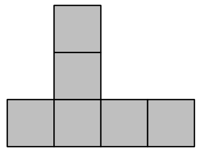
REMARK

Note the useful check: with $x = \frac{2}{3}$, the side lengths of the triangle QOP are given by $OQ = 1$, $QP = \frac{4}{3}$ and $OP = \frac{5}{3}$. These are in the ratio 3 : 4 : 5, which (using the converse of Pythagoras' Theorem) confirms that $\angle OQP$ is a right angle.

20. The diagram shows six squares with sides of length 2 placed edge-to-edge.

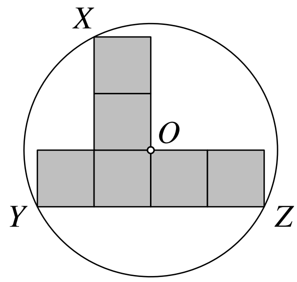
What is the radius of the smallest circle containing all six squares?

A $2\sqrt{5}$ B $2\sqrt{6}$ C 5 D $\sqrt{26}$ E $2\sqrt{7}$



Solution

A We let O, X, Y and Z be the points shown in the figure on the right. By Pythagoras' Theorem the points X, Y and Z are all at a distance $2\sqrt{5}$ from O. It is straightforward to check that the circle with centre O and radius $2\sqrt{5}$ contains all six squares. So the radius of the smallest circle containing all six squares is at most $2\sqrt{5}$.



In the context of the SMC it is sufficient to note that $2\sqrt{5}$ is the smallest of the given options, and so must be the correct answer.

However, to give a complete mathematical solution we need to show that no circle with a smaller radius will do.

We have seen that the circle that goes through X, Y and Z includes all six squares. Since any circle that includes all six squares must include these three points, to complete the proof all we need show is that the circle that goes through X, Y and Z is the smallest circle which includes all these three points.

The circle that goes through the vertices of a triangle is called the *circumcircle* of the triangle.

We note first that XYZ is an acute-angled triangle. You are asked to check this in Problem 20.1.

It will therefore be sufficient to prove the general result:

If PQR is an acute-angled triangle then the smallest circle that includes PQR is its circumcircle.

At first sight this may seem obvious. However, the condition that the triangle is acute-angled is essential.

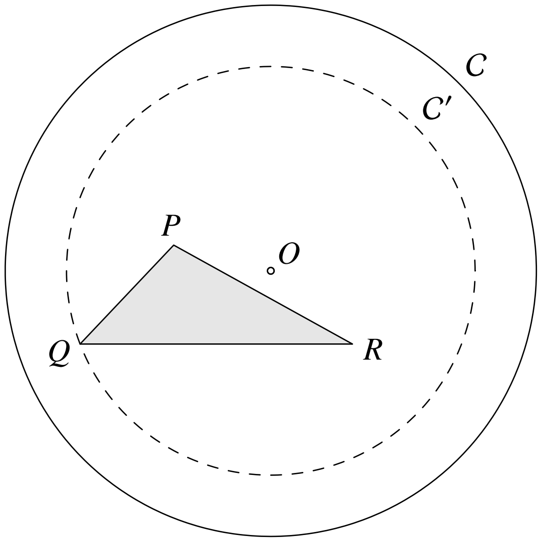
You are asked to show in Problem 20.5 that the circumcircle of a triangle with an obtuse angle is *not* the smallest circle that includes the triangle.

So things are a little more subtle than might first appear. Somewhere our argument must use the fact that we are dealing with a triangle all of whose angles are acute. Watch out for the step where this is used.

Suppose to begin with that the circle C with centre O includes the triangle PQR as in the figure on the right.

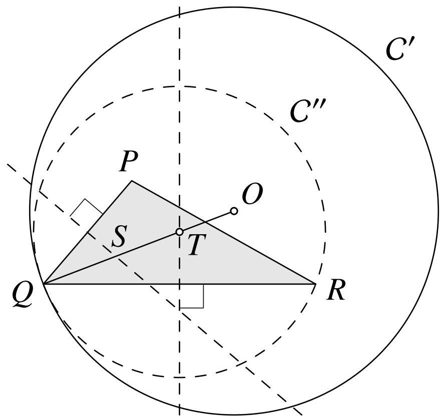
It is easy to see that if none of the points P, Q and R lies on the circle, then we can contract C to obtain a smaller circle, C', also with centre O, which includes the triangle PQR and goes through at least one of the vertices of the triangle.

You are asked in Problem 20.2 to show how this smaller circle may be drawn.



Now consider this smaller circle, C', with centre O, which includes the triangle PQR, and goes through one of its vertices, say Q, but not the other two, as in the figure on the right below.

We can contract the circle to obtain a circle, C'', that still goes through Q, but also goes through another of the vertices of the triangle PQR. This is fairly obvious but we show precisely how the circle C'' may be constructed.



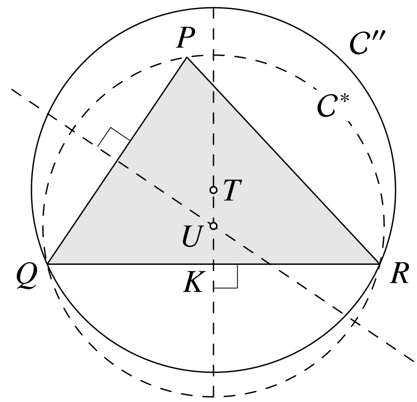
The points on the perpendicular bisector of QP are equidistant from Q and P. The points on one side of it are closer to Q than to P, and the points on the other side are closer to P than to Q. As P is inside the circle C', O is closer to P than to Q. Therefore O lies on the opposite side of the perpendicular bisector of PQ from Q. Hence the perpendicular bisector of PQ meets OQ at some point, say S, that lies between O and Q.

Similarly, the perpendicular bisector of QR meets OQ at some point say T, that lies between O and Q.

Suppose that $OT < OS$. Then, as $TQ = TR$ and $TP < TQ$, the circle C'', with centre T and radius TQ, goes through Q and R and includes the point P. Since $TQ < OQ$, the circle C'' is smaller than the circle C'.

In the case where $OT < OS$, the argument is similar and we end up with a circle, C'', smaller than C', that goes through P and Q and includes R. If $OT = OS$, the points S and T coincide and C'' is the circumcircle of the triangle PQR.

To complete the proof we now show that if PQR is an acute-angled triangle and C'' is a circle which includes the triangle and goes through the vertices Q and R but not P, then the circumcircle, C^*, of PQR has a radius which is smaller than that of C''.



Let T be the centre of the circle C''. Since C'' goes through Q and R, it follows that T lies on the perpendicular bisector of QR. Since P lies inside C'', we have $TP < TQ$.

Let K be the midpoint of QR. Since $\angle QPR$ is acute, we have $KQ < KP$ (You are asked to prove this in Problem 20.3.)

It follows that the points T and K are on opposite sides of the perpendicular bisector of PQ, and therefore the point U, where the perpendicular bisector of PQ meets KT, lies between T and K.

The point U is the centre of the circumcircle of triangle PQR. As U is between T and K, $UK < TK$, and so $\sqrt{QK^2 + UK^2} < \sqrt{QK^2 + TK^2}$. Therefore, by Pythagoras' Theorem, $QU < QT$. So the radius of the circumcircle, C^*, of triangle PQR is less than the radius of the circle C''.

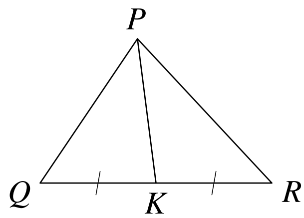
It follows from this general result that the circumcircle of the acute-angled triangle XYZ is the smallest circle which includes the points X, Y and Z. Therefore it is the smallest circle that contains all six squares. We have already seen that this circumcircle has centre O and radius $2\sqrt{5}$. So the smallest circle which contains all six squares has radius $2\sqrt{5}$.

For investigation

20.1 Check that the side lengths of the triangle XYZ are $\sqrt{10}$, 4 and $3\sqrt{2}$ and verify that it is an acute-angled triangle.

20.2 Suppose that the points P, Q and R are all inside the circle C with centre O. Show how to find the radius of a circle C' with centre O, which includes the triangle PQR and which passes through at least one of its vertices.

20.3 Prove that if, in the triangle QPR, $\angle QPR$ is acute and K is the midpoint of QR, then $KQ < KP$.



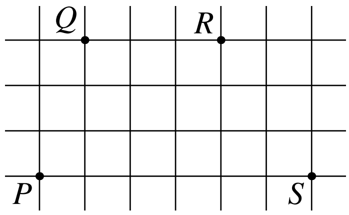
20.4 In the above solution we have twice used the fact that:

> The point where the perpendicular bisectors of two of the sides of a triangle meet is the centre of the circumcircle of the triangle.

Explain why this is true.

20.5 Let PQR be a triangle in which $\angle QPR$ is obtuse. Find the smallest circle that includes the triangle, and show that this circle is not the circumcircle of the triangle.

21. Fiona wants to draw a 2-dimensional shape whose perimeter passes through all of the points P, Q, R and S on the grid of squares shown.



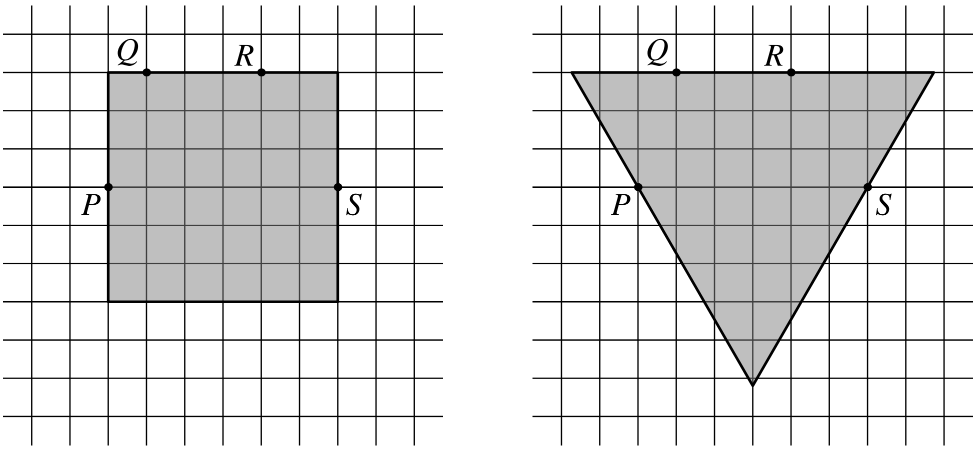
Which of the following can she draw?

(i) A circle
(ii) An equilateral triangle
(iii) A square

A only (i) and (ii) B only (ii) and (iii) C only (i) and (iii)
D all of (i), (ii) and (iii) E none of (i), (ii) and (iii)

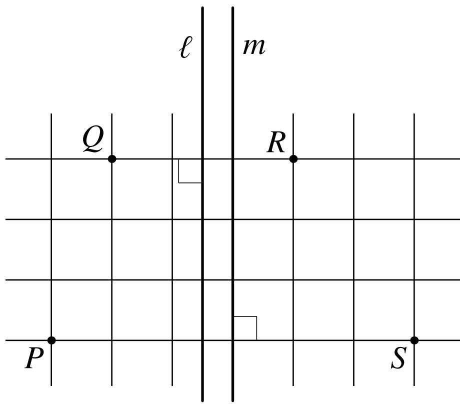
Solution

B The figure on the left below shows that it is possible to draw a square through the points P, Q, R and S. The figure on the right shows that it is also possible to draw an equilateral triangle through these points.



We now show that it is not possible to draw a circle through these four points.

The centre of each circle through the points Q and R is equidistant from Q and R and hence lies on the perpendicular bisector, l, of QR.



Similarly the centre of each circle through the points P and S lies on the perpendicular bisector, m, of PS.

The lines l and m are each perpendicular to the horizontal lines of the grid and so they are parallel. So there is no point which is on both lines.

It follows that there is no point equidistant from P, Q, R and S. Therefore there is no circle which goes through these four points.

It follows that B is the correct option.

22. A bag contains m blue and n yellow marbles. One marble is selected at random from the bag and its colour is noted. It is then returned to the bag along with k other marbles of the same colour. A second marble is now selected at random from the bag.

What is the probability that the second marble is blue?

A $\frac{m}{m+n}$ B $\frac{n}{m+n}$ C $\frac{m}{m+n+k}$ D $\frac{m+k}{m+n+k}$ E $\frac{m+n}{m+n+k}$

Solution

A In the context of the SMC it is sufficient to eliminate those options which cannot be correct for all possible values of m, n and k.

The probability that the second marble selected is blue is 0 if $m = 0$, and is 1 if $n = 0$. Only the formula given in option A meets both these requirements. So, assuming that one of the options is correct, it must be option A.

However, for a complete mathematical solution we need to give an argument to show that the formula given in option A is correct. We do this as follows.

The probability that the first marble selected is blue is $\frac{m}{m+n}$. If the first marble chosen is blue, it and k other blue marbles are put in the bag, which now contains $m+n+k$ marbles of which $m+k$ are blue. So the probability now that the second marble that is selected is blue is $\frac{m+k}{m+n+k}$. Therefore the probability that the first marble selected is blue and the second marble selected is blue is $\left(\frac{m}{m+n}\right)\left(\frac{m+k}{m+n+k}\right)$.

The probability that the first marble selected is yellow is $\frac{n}{m+n}$. If the first marble selected is yellow, it and k other yellow marbles are placed in the bag, which now contains $m+n+k$ marbles of which m are blue. So the probability now that the second marble that is selected is blue is $\frac{m}{m+n+k}$. Therefore the probability that the first marble selected is yellow and the second marble that is selected is blue is $\left(\frac{n}{m+n}\right)\left(\frac{m}{m+n+k}\right)$.

Hence the overall probability that the second marble selected is blue is

$$\left(\frac{m}{m+n}\right)\left(\frac{m+k}{m+n+k}\right) + \left(\frac{n}{m+n}\right)\left(\frac{m}{m+n+k}\right) = \frac{m(m+k)+mn}{(m+n)(m+n+k)}$$

$$= \frac{m(m+n+k)}{(m+n)(m+n+k)} = \frac{m}{m+n}.$$

Remark

Notice that this probability is independent of the value of k and is the same as the probability that the first marble selected is blue. We have been unable to find an explanation of why this is so other than the calculation given in the solution. If you have a more straightforward way to explain this, please let us know!

23. Which of the following have no real solutions?

(i) $2x < 2^x < x^2$ (ii) $x^2 < 2x < 2^x$ (iii) $2^x < x^2 < 2x$ (iv) $x^2 < 2^x < 2x$
(v) $2^x < 2x < x^2$ (vi) $2x < x^2 < 2^x$

A (i) and (iii) B (i) and (iv) C (ii) and (iv) D (ii) and (v)
E (iii) and (v)

Solution

E We can deduce from the wording of the question that four of the given inequalities have solutions and two do not.

In the context of the SMC it is sufficient to find numerical solutions for four of the inequalities, because this will eliminate all but one of the given options.

After trying various values for x it turns out that we need only consider the values -1, $\frac{1}{2}$, $\frac{3}{2}$ and 5 (other choices will also work).

When $x = -1$, we have $2x = -2$, $2^x = \frac{1}{2}$ and $x^2 = 1$. So in this case $2x < 2^x < x^2$. Therefore (i) has a solution.

When $x = \frac{1}{2}$, we have $x^2 = \frac{1}{4}$, $2x = 1$ and $2^x = \sqrt{2}$. So in this case $x^2 < 2x < 2^x$. Therefore (ii) has a solution.

When $x = \frac{3}{2}$, we have $x^2 = \frac{9}{4}$, $2^x = 2\sqrt{2}$ and $2x = 3$. So in this case $x^2 < 2^x < 2x$. Therefore (iv) has a solution.

Finally, when $x = 5$, we have $2x = 10$, $x^2 = 25$ and $2^x = 32$. So in this case $2x < x^2 < 2^x$. Therefore (vi) has a solution.

We can therefore deduce that it is inequalities (iii) and (v) that do not have solutions.

However, a complete mathematical solution requires us to prove that (iii) and (v) have no solutions.

One approach might be to draw the graphs of the functions, and you are asked to do this in Problem 23.2 below. However, this leaves open the question as to how we can be sure that the information derived from the graphs is correct. After all, we cannot draw a graph which shows the full range of values of x. So this approach wouldn't count as a fully rigorous argument.

A fully rigorous argument which does not rely on drawing a graph uses ideas from advanced calculus that are not normally met until the first year of a university course. We therefore do not give this argument here, but Problems 23.3 and 23.4 will begin to lead you in the right direction.

For investigation

23.1 In the above solution, in showing that $x = \frac{3}{2}$ is a solution of inequality (iv), we used the fact that $\frac{9}{4} < 2\sqrt{2} < 3$. Show, *without using a calculator*, that this is correct.

23.2 Draw the graphs of the curves given by $y = 2x$, $y = x^2$ and $y = 2^x$.

It is difficult to draw these graphs by hand accurately enough. So it is best to use a computer graph plotter. This will indicate that there are no values of x for which either of the inequalities $2^x < x^2 < 2x$ or $2^x < 2x < x^2$ holds. However, as mentioned above, this not a full mathematical solution without an argument to show that the information derived from the graphs is correct.

23.3 Find the values of x for which $x^2 = 2x$ and for which $x^2 < 2x$.

23.4 Find all the solutions of the equations $2^x = 2x$ and $2^x = x^2$. In each case, how can you be sure that you have found *all* the solutions?

24. Which of the following is smallest?

A $10 - 3\sqrt{11}$ B $8 - 3\sqrt{7}$ C $5 - 2\sqrt{6}$ D $9 - 4\sqrt{5}$ E $7 - 4\sqrt{3}$

Solution

A We first note that $10 - 3\sqrt{11} = \sqrt{100} - \sqrt{99}$, $8 - 3\sqrt{7} = \sqrt{64} - \sqrt{63}$, $5 - 2\sqrt{6} = \sqrt{25} - \sqrt{24}$, $9 - 4\sqrt{5} = \sqrt{81} - \sqrt{80}$, and $7 - 4\sqrt{3} = \sqrt{49} - \sqrt{48}$. So the options are given by the formula $\sqrt{n+1} - \sqrt{n}$, for $n = 99, 63, 24, 80$ and 48, respectively.

If we put $x = \sqrt{n+1}$ and $y = \sqrt{n}$ in the *difference of two squares* identity $(x - y)(x + y) = x^2 - y^2$, we obtain

$$(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n}) = (n+1) - n = 1.$$

From this it follows that

$$\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

Since $\sqrt{n+1} + \sqrt{n}$ increases as n increases, it follows that the larger the value of n the smaller is the value of $\sqrt{n+1} - \sqrt{n}$. Therefore the smallest of the given options corresponds to the largest value of n. So $10 - 3\sqrt{11}$ is the smallest of the given options.

For investigation

24.1 Arrange the numbers given as the options in Question 24 in order of magnitude, with the smallest number first, *without using a calculator*.

24.2 Arrange the following numbers in order of magnitude, with the smallest number first (again *without using a calculator*, of course).

$$6 - 3\sqrt{3},\ 7 - 2\sqrt{10},\ 9 - 6\sqrt{2},\ 11 - 4\sqrt{7},\ 13 - 4\sqrt{10}.$$

25. Figure 1 shows a tile in the form of a trapezium, where $\alpha = 83\frac{1}{3}°$. Several copies of the tile are placed together to form a symmetrical pattern, part of which is shown in Figure 2. The outer border of the complete pattern is a regular 'star polygon'. Figure 3 shows an example of a regular 'star polygon'.

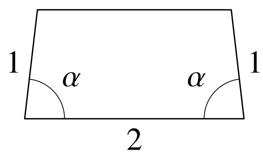


Figure 1

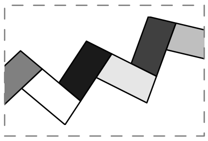


Figure 2

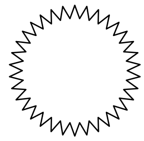


Figure 3

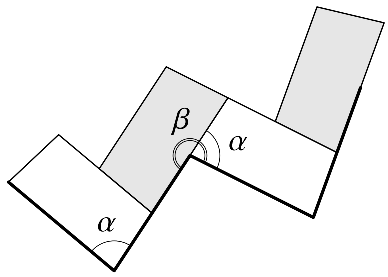
How many tiles are there in the complete pattern?

A 48 B 54 C 60 D 66 E 72

Solution

B The figure on the right shows part of the complete pattern. The heavier lines form part of the border of the 'star polygon'.

The complete pattern is made up of an even number of tiles. Let the number of tiles be $2n$, where n is a positive integer.



The 'star polygon' has $2n$ edges and so it has $2n$ internal angles. From the figure we see that n of these angles have size α and n of them have size β, where $\beta = \alpha + 180°$. Therefore the sum of the internal angles of the 'star polygon' is $n\alpha + n(\alpha + 180°)$, that is, $n(2\alpha + 180°)$. Since $\alpha = 83\frac{1}{3}°$, $2\alpha + 180° = \frac{1040°}{3}$. So the sum of the internal angles is $n \times \frac{1040°}{3}$.

On the other hand, the sum of the internal angles of a polygon with $2n$ edges is $(2n-2)\times 180°$.

The two expressions we have obtained for the sum of the internal angles are equal, and so we have

$$(2n - 2) \times 180 = n \times \frac{1040}{3}.$$

This last equation may be rearranged to give

$$\left(360 - \frac{1040}{3}\right)n = 360,$$

that is,

$$\frac{40}{3}n = 360,$$

from which it follows that

$$\begin{aligned} n &= \frac{3}{40} \times 360 \\ &= 27. \end{aligned}$$

We deduce that the 'star polygon' has $2n = 54$ sides.

For an alternative method, see Problem 25.4.

For investigation

25.1 The solution uses the fact that the sum of the internal angles of a polygon with $2n$ edges is $(2n - 2) \times 180°$. Show why this is the case.

25.2 Suppose that $\alpha = 80°$. How many edges does the 'star polygon' have in this case?

25.3 Suppose that the 'star polygon' has 72 edges. What is the value of α in this case?

25.4 In this problem we consider an alternative method for answering Question 25. Consider moving anticlockwise around the perimeter of the 'star polygon'. At each vertex you either turn anticlockwise through the angle $(180° - \alpha)$ or clockwise through the angle α.

(a) Suppose that the 'star polygon' has $2n$ edges. What is the total anticlockwise angle that you turn through as you move anticlockwise around it?

(b) Since in going completely round the 'star polygon' anticlockwise you turn through an angle 360°, what value does this give for n?

Remark

This problem was inspired by some floor tiling in the Church on Spilled Blood in St Petersburg. This church, also called the Church of the Saviour on Spilled Blood or the Cathedral of the Resurrection of Christ, is in St Petersburg in Russia.