Solutions to the European Kangaroo Pink Paper 2015

1. **C** The units digits of 2015^2, 2015^0, 2015^1, 2015^5 are 5, 1, 5, 5, which add to 16. Thus, the units digit of the sum is 6.

2. **B** If the semicircle is cut into two quarter-circles, these can be placed next to the other shaded region to fill up half the square. Hence the shaded area is half of the area of the square, namely $\frac{1}{2}a^2$.

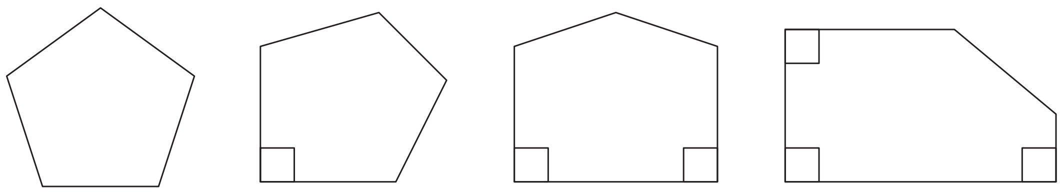
3. **A** Points which are at most 5 m from the tree lie on or inside a circle of radius 5 m with its centre at the tree. However, not all of the inside of the circle will be shaded because the treasure is at least 5 m from the hedge, so we should have an unshaded rectangular strip next to the hedge. This leaves the shaded region in A.

4. **E** Anya pays 80p out of a total of 80p + 50p + 20p = 150p. So if the biscuits had been divided in the same ratio as the payments, she would have received $\frac{80}{150} \times 30 = 16$ biscuits. So she would have received $16 - 10 = 6$ more biscuits.

5. **E** There are three children who like both subjects, leaving 30 children to be shared in the ratio 2:1. Hence 20 like only Computer Science, 10 like only PE, 3 like both. The total number who like Computer Science is 20 + 3 = 23.

6. **E** Using the index law $a^{mn} = (a^m)^n$, we see $2^9 = (2^3)^3$, a cube; $3^{10} = (3^5)^2$, a square; $4^{11} = (2^2)^{11} = (2^{11})^2$, a square; and $5^{12} = (5^4)^3$, a cube. This leaves 6^{13} which is neither a square nor a cube since 6 is neither a square nor a cube, and 13 is not divisible by 2 nor by 3.

7. **C** The diagrams below show that it is possible to find pentagons with 0, 1, 2, 3 right angles. The angles in a pentagon add to 540°, so with 4 right angles, the fifth angle would be $540° - 4 \times 90° = 180°$, which would make the shape flat at that vertex, thus creating a quadrilateral. Also, a pentagon with 5 right angles is not possible, because they wouldn't add up to 540°.



8. **B** We will show that the word "YES" appears exactly three times, giving the probability 3/6 or 1/2. Firstly note that "YES" appears twice on the second die. The third die also shows "YES" and this cannot be the same as either "YES" on the second die: Under the first "YES" is "MAYBE", but on the third die the word "NO" appears below it; to the left of the second "YES" is "MAYBE", but to the left of the "YES" on the third die is "NO". Hence "YES" appears at least three times. However, it appears at most three times because there are two occurrences of "NO" shown in the third die, and one "MAYBE" in the second die. The first die has not been used in the above argument, but is consistent with the faces showing "YES" three times, "NO" twice, and "MAYBE" once.

9. **C** The shortest routes consist of two diagonals (right and down) each of length $\sqrt{2}$, and two sides of length 1, giving a total length $2 + 2\sqrt{2}$.

10. **C** The aliens say they can see 8, 7, 5 ears respectively, but each ear has been seen by two aliens so is counted twice. Hence the total number of ears is $\frac{1}{2}(8 + 7 + 5) = 10$ ears. Each alien can see all ten ears, except its own. Timi sees 5 ears, so has $10 - 5 = 5$ ears.

11. **D** Let l be the length of KG. Then $FK = 3l$, and the sides of the square $FGHI$ are each $4l$.

Since the area of the square is 80, we have $(4l)^2 = 80$, which is $16l^2 = 80$; hence $l^2 = 5$.

By Pythagoras' Theorem, $JK^2 = FJ^2 + FK^2 = l^2 + (3l)^2 = 10l^2$.

The shaded area is half the area of the square $JKLM$, i.e. half of $JK^2 = \frac{1}{2} \times 10l^2 = 5l^2 = 5 \times 5 = 25$.

12. **C** The prime factor decomposition of 2015 is $5 \times 13 \times 31$, so the only possible pairs of ages are 1×2015, 5×403, 13×155, 31×65. The only realistic pair of ages is 65 and 31, with a difference of 34.

13. **A** From the first picture, we can see:

From the right scale:	$Z > Y$	(1)
From the left scale:	$X > W$	(2)
From the large scale:	$Y + Z > W + X$	(3)

It follows from (1), (2) and (3) that $Z + Z > Y + Z > W + X > W + W$, so $2Z > 2W$, and hence $Z > W \ldots$ (4).

We can show that most swaps give a contradiction of these inequalities:

Firstly, suppose that weight Z doesn't move. Then there are three possible swaps:

X and Y swap: then in the second picture we must have $Z < X$ and $Y < W$, which add to give $Y + Z < W + X$, contradicting (3).

Y and W swap: then we would have $Z < W$, contradicting (4).

X and W swap: then we would have $X < W$, contradicting (2).

Hence Z must swap, which can happen in three ways:

Z and Y swap: then we would have $Z < Y$, contradicting (1).

Z and X swap: then we would have $Z < W$, contradicting (4).

Z and W swap: This is the only possibility left, and it can work when the weights W, X, Y, Z are 1, 2, 3, 4. The swap would give 4, 2, 3, 1 which matches the picture on the right. However, this does depend on the values of W, X, Y, Z; it would not work for 1, 5, 3, 4.

14. **B** Suppose the quadratic has roots p and q. So it factorises as $(x - p)(x - q)$. This expands to give $x^2 - (p + q)x + pq$. Comparing this with $x^2 - 85x + c$ shows that $p + q = 85$ and $pq = c$. It follows that one of p, q is even and one is odd. The only even prime is 2 so the primes p, q are 2 and 83. Therefore $c = 2 \times 83 = 166$, and therefore has digit sum 13.

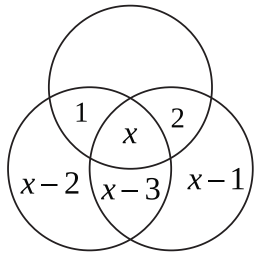
15. **E** The first digit can be anything from 1 to 9. Where possible, we can reduce this by three or increase it by three to get the next digit, giving the following possibilities for the first two digits: 14, 25, 30, 36, 41, 47, 52, 58, 63, 69, 74, 85, 96. Repeating for the third digit, we obtain the possibilities in numerical order: 141, 147, 252, 258, 303, 363, 369, 414, 474, 525, 585, 630, 636, 696, 741, 747, 852, 858, 963, 969, which is 20 options.

16. **E** Since 21 is not prime, it cannot give a contradiction to the statement because n must be prime.

The primes 11, 19, 29 don't give a contradiction because exactly one of $n - 2$, $n + 2$ is a prime for each of them: 11 (9 is not prime, 13 is prime), 19 (17 is prime, 21 is not prime), 29 (27 is not prime, 31 is prime).

However for 37, which is prime, this does give a contradiction.

17. **E** Let x be the number in the central region. Since this is the sum of its three neighbouring regions which include 1 and 2, the region below it must contain $x - 3$. The bottom right region then contains $(x - 3) + 2 = x - 1$. The bottom left region then contains $(x - 3) + 1 = x - 2$. But the number in the bottom central region can now be evaluated in two ways, firstly as $x - 3$, but also as the sum of its neighbours, x, $x - 1$, $x - 2$. Hence $x - 3 = x + (x - 1) + (x - 2)$, giving $x - 3 = 3x - 3$. So $x = 3x$, giving $2x = 0$ and so $x = 0$.



18. **B** Petra can arrange the dictionaries in six ways (3 choices for the first dictionary, 2 choices for the second dictionary, 1 choice for the third, giving $3 \times 2 \times 1 = 6$ ways). The novels can be arranged in two ways. Since the novels could be on the left of the dictionaries, or on the right, we have a total of $6 \times 2 \times 2 = 24$ ways to arrange the books.

19. **C** Since $2^7 = 128$ is larger than 100, the only powers we can choose from are the first seven powers: 2^0 to 2^6, i.e. 1, 2, 4, 8, 16, 32, 64. The sum of all seven is 127. We need to eliminate one of these, and be left with a total under 100, so the only possibilities to remove would be 32 or 64. Hence there are two options:

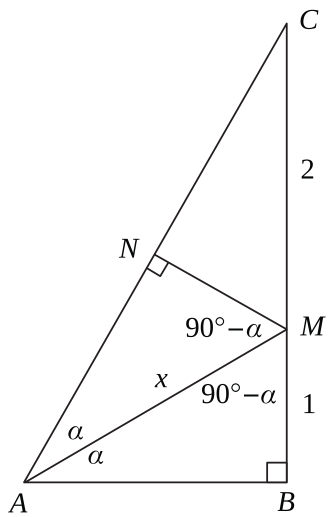
$1 + 2 + 4 + 8 + 16 + 32 = 63$, and $1 + 2 + 4 + 8 + 16 + 64 = 95$.

20. **D** In the triangle on the left, the unshaded triangle is similar to triangle FGH, and is obtained from it by a scale factor of $\frac{4}{5}$. Hence its area is $\left(\frac{4}{5}\right)^2 = \frac{16}{25}$ of the area of FGH. The shaded area is therefore $\frac{9}{25}$ of the area of FGH. Hence $HY : HF = 3 : 5$ and so $HY : YF = 3 : 2$.

21. **C** Label the vertices of the right-angled triangle as A, B, C, with angle $ABC = 90°$. Let M be the point where the angle bisector of CAB meets the side BC. Let N be the point where the perpendicular from M meets the side AC. Let x be the length of the bisector AM.

Then triangles ABM and ANM are congruent (they have all three angles the same, and one corresponding side AM in common). Thus $MN = 1$.

If we reflect triangle MNC in the line NC, then we have an equilateral triangle with all sides equal to length 2. Hence $\angle NCM = 30°$ (half of 60°). Thus $\angle CAB = 180° - 90° - 30° = 60°$, and $\alpha = 30°$. Then triangles MNC and MNA are congruent (all angles the same, and common length MN), so $AM = MC = 2$.



An alternative solution can be obtained using the Angle Bisector Theorem. This gives us $AB : AC = BM : CM = 1 : 2$. Suppose, then, that $AB = x$ and $AC = 2x$. By Pythagoras' Theorem applied to triangle ABC, $(2x)^2 = x^2 + 3^2$. Therefore $x^2 = 3$. Therefore, by Pythagoras' Theorem applied to triangle MBA, $AM^2 = x^2 + 1^2 = 3 + 1 = 4$. Therefore $AM = \sqrt{4} = 2$.

22. A Since the three digits a, b, c are different and $\overline{ab} < \overline{bc} < \overline{ca}$, it is necessary that $a < b < c$. But this condition is also sufficient. Also note that none of a, b, c are zero because they each represent the tens digit of a two-digit number.

There are 9 ways to pick a non-zero digit, 8 ways to pick a second (different) digit, and 7 ways to pick a third digit. These digits can be arranged in $3 \times 2 \times 1 = 6$ ways, but only one of these will be in ascending order (we need $a < b < c$). Hence there are $(9 \times 8 \times 7) \div 6 = 84$ ways to choose the digits a, b, c.

23. B The sum of $1, 2, \ldots n$ is $\frac{1}{2}n(n + 1)$. So the smallest the sum could be if one number were deleted would be $\frac{1}{2}n(n+1) - n = \frac{1}{2}\left(n^2 + n - 2n\right) = \frac{1}{2}n(n-1)$. Thus the mean of any $n - 1$ of these numbers is at least $\frac{1}{2}n$. So for the mean to be 4.75, we must have $n \leqslant 9$.

Similarly, the largest the sum could be with one number deleted would be $\frac{1}{2}n(n+1) - 1 = \frac{1}{2}\left(n^2 + n - 2\right) = \frac{1}{2}(n-1)(n+2)$. Thus the mean of any $n - 1$ of the numbers is at most $\frac{1}{2}(n + 2)$. So we must have $\frac{1}{2}(n + 2) \geqslant 4.75$ and so $n \geqslant 8$.

Since $4.75 = 19/4$ and the mean is obtained by dividing by $n - 1$, we require $n - 1$ to be a multiple of 4. So $n = 9$. The sum of $1, \ldots, 9$ is 45 and we need the 8 numbers used to have a mean of 4.75. So their total is 38 and hence 7 is the number to be eliminated.

24. C It is certainly possible to underline two numbers. In the list below both 1 and -1 are equal to the product of the other nine numbers $1, -1, -2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}, 5, \frac{1}{5}$.

However, it is not possible to underline three numbers. For supposing it was possible to underline the numbers a, b, c. Let N be the product of the other seven numbers. Then we have

$$a = b \times c \times N \ldots (1) \qquad b = a \times c \times N \ldots (2) \qquad c = a \times b \times N \ldots (3)$$

Substituting (1) into (2) gives $b = (b \times c \times N) \times c \times N = b \times c^2 \times N^2$ so $c^2 \times N^2 = 1$. Thus $c \times N = 1$ or $c \times N = -1$. But if $c \times N = 1$, then (1) becomes $a = b$, contradicting that a, b are distinct. Hence $c \times N = -1$, and (1) becomes $a = -b$. Substituting this into (3) gives $c = (-b) \times b \times N$. Multiplying both sides by c gives $c^2 = -b^2 \times c \times N = -b^2 \times (-1) = b^2$. Hence either $c = b$ (contradicting that they are distinct) or $c = -b = a$ (contradicting that a, c are distinct).

Hence there is no way that a, b, c can be distinct numbers and also underlined. (If the numbers are allowed to be the same, it is possible to underline them all by choosing the numbers to be all equal to one).

25. B Suppose the first of the two special points mentioned has a points to its left, and b points to its right. Then the number of line segments it lies on is $a \times b$ (a choices for the left end, b choices for the right end). Also the number of points will be $a + b + 1$. Similarly if the second point has c points to its left, and d to the right, then the number of line segments it lies on is $c \times d$, and the number of points is $c + d + 1$.

Hence we need to find integers a, b, c, d such that $ab = 80$, $cd = 90$, $a + b + 1 = c + d + 1$ (or more simply $a + b = c + d$).

The factor pairs of 80 (and hence the possible values of a, b) are 1×80, 2×40, 4×20, 5×16, 8×10.

The factor pairs of 90 (and hence possible values of c, d) are 1×90, 2×45, 3×30, 5×18, 6×15, 9×10.

The only pairs for which we have $a + b = c + d$ are 5, 16 and 6, 15. Since both of these add to 21, the number of points must be $a + b + 1 = 21 + 1 = 22$.