Pink Kangaroo

a member of the Association Kangourou sans Frontières

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

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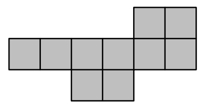
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
B	B	B	C	B	E	B	A	B	A	C	E	D	A	D	B	C	E	D	C	C	C	A	E	B

1. The diagram shows a shape made from ten squares of side-length 1 cm, joined edge to edge.



What is the length of its perimeter, in centimetres?

A 14 B 18 C 30 D 32 E 40

Solution **B**

Counting the edges of the squares around the shape gives a perimeter of 18 cm.

2. When the answers to the following calculations are put in order from smallest to largest, which will be in the middle?

A 1 + 23456 B 12 + 3456 C 123 + 456 D 1234 + 56 E 12345 + 6

Solution **B**

The answers are 23457, 3468, 579, 1290 and 12351 respectively, so in ascending order the middle one is B.

3. In the calculations shown, each letter stands for a digit. They are used to make some two-digit numbers. The two numbers on the left have a total of 79.

What is the total of the four numbers on the right?

A 79 B 158 C 869 D 1418 E 7979

$$\begin{array}{r} \text{J K} \\ +\ \text{L M} \\ \hline 7\ 9 \end{array} \qquad \begin{array}{r} \text{J M} \\ +\ \text{L M} \\ +\ \text{J K} \\ +\ \text{L K} \\ \hline ? \end{array}$$

Solution **B**

The numbers on the left use the digits M and K in the Units column, and the digits J and L in the Tens column. The numbers on the right use the digits M and K twice in the Units columns, and the digits J and L twice in the Tens column, so the total is exactly twice that of the numbers on the left. Twice 79 is 158, answer B.

4. The sum of four consecutive integers is 2. What is the least of these integers?

A -3 B -2 C -1 D 0 E 1

Solution **C**

The four integers cannot all be non-positive since the total is positive. Also, they cannot all be non-negative since the smallest possible total would then be $0 + 1 + 2 + 3 = 6$. So, they must have at least one negative integer, and at least one positive integer, and hence will include -1, 0 and 1. Since these have a total of 0, the fourth number must be 2. Thus the least integer is -1.

5. The years 2020 and 1717 both consist of a repeated two-digit number.

How many years after 2020 will it be until the next year which has this property?

A 20 B 101 C 120 D 121 E 202

Solution **B**

The next year with this property is 2121 which is 101 years after 2020.

6. Mary had ten pieces of paper. Some of them were squares, and the rest were triangles. She cut three squares diagonally from corner to corner. She then found that the total number of vertices of the 13 pieces of paper was 42.

How many triangles did she have before making the cuts?

A 8 B 7 C 6 D 5 E 4

Solution **E**

Let s be the number of squares, and t the number of triangles that Mary started with. The number of vertices was $4s + 3t$. Also, there were 10 pieces of paper, so

$$s + t = 10. \qquad [1]$$

When she cuts a square diagonally to create two triangles, she increases the number of vertices by 2 (from 4 to 6). Hence before she cut the three squares, she had $42 - 3 \times 2 = 36$ vertices. Therefore

$$4s + 3t = 36. \qquad [2]$$

Subtracting three times equation [1] from equation [2] gives $s = 6$. Hence $t = 4$.

7. The positive integers a, b, c, d satisfy the equation $ab = 2cd$.

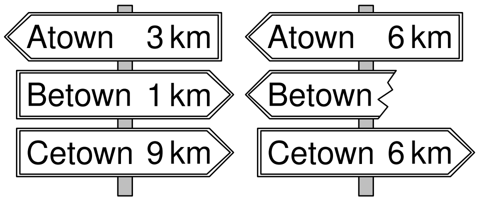
Which of the following numbers could not be the value of the product $abcd$?

A 50 B 100 C 200 D 450 E 800

Solution **B**

Since $ab = 2cd$, the product $abcd = 2cd \times cd = 2(cd)^2$, hence it must be twice a perfect square. This is true for all the options, except 100 since $100 = 2 \times 50$ but 50 is not a perfect square. [$50 = 2 \times 5^2$; $200 = 2 \times 10^2$; $450 = 2 \times 15^2$; and $800 = 2 \times 20^2$.]

8. The shortest path from Atown to Cetown runs through Betown. Two of the signposts that can be seen on this path are shown, but one of them is broken and a number missing.



What distance was written on the broken sign?

A 2 km B 3 km C 4 km D 5 km E 6 km

Solution **A**

The first signpost shows that Betown is 4 km from Atown. The second signpost is 6 km from Atown, so must be 2 km from Betown.

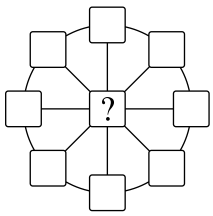
9. An isosceles triangle has a side of length 20 cm. Of the remaining two side-lengths, one is equal to two-fifths of the other. What is the length of the perimeter of this triangle?

A 36 cm B 48 cm C 60 cm D 90 cm E 120 cm

Solution **B**

The triangle is isosceles so has a pair of equal sides. Since the two unknown sides are not equal, they cannot be the pair of equal sides, and hence the 20 cm side must be one of the pair of equal sides. The base is then two-fifths of 20 cm, namely 8 cm. The perimeter is $20 + 20 + 8 = 48$ cm.

10. Freda wants to write a number in each of the nine cells of this figure so that the sum of the three numbers on each diameter is 13 and the sum of the eight numbers on the circumference is 40.



What number must be written in the central cell?

A 3 B 5 C 8 D 10 E 12

Solution **A**

Each diameter has the same sum and contains the central cell, so the pair at the end of each diameter must have the same sum. These four pairs have sum 40, so each pair must have sum 10. Since each diameter has sum 13, the central number must be 3.

11. Masha put a multiplication sign between the second and third digits of the number 2020 and noted that the resulting product 20×20 was a square number.

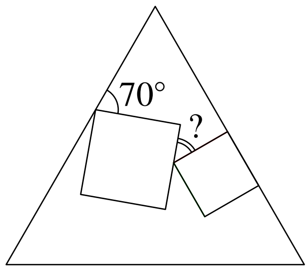
How many integers between 2010 and 2099 (including 2020) have the same property?

A 1 B 2 C 3 D 4 E 5

Solution **C**

Each number begins with 20, and $20 = 5 \times 2^2$. Hence, to make a square product, the last two digits must make a number which is a product of 5 and a square number. The possibilities between 10 and 99 are $5 \times 2^2 = 20$, $5 \times 3^2 = 45$ and $5 \times 4^2 = 80$. Therefore there are three possible numbers: 2020, 2045, 2080.

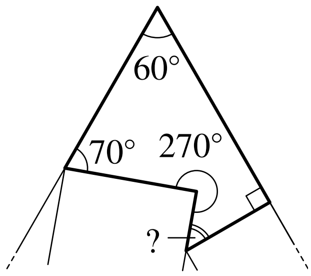
12. Two squares of different sizes are drawn inside an equilateral triangle. One side of one of these squares lies on one of the sides of the triangle as shown. What is the size of the angle marked by the question mark?



A 25° B 30° C 35° D 45° E 50°

Solution **E**

On the diagram is a pentagon outlined, and four of its angles are known. The sum of the angles in a pentagon is 540°. The missing angle is then 540° −270° − 70° − 60° − 90° = 50°.



13. Luca began a 520 km trip by car with 14 litres of fuel in the car tank. His car consumes 1 litre of fuel per 10 km. After driving 55 km, he saw a road sign showing the distances from that point to five petrol stations ahead on the road. These distances are 35 km, 45 km, 55 km, 75 km and 95 km. The capacity of the car's fuel tank is 40 litres and Luca wants to stop just once to fill the tank.

How far is the petrol station that he should stop at?

A 35 km B 45 km C 55 km D 75 km E 95 km

Solution **D**

Luca starts with 14 litres of fuel, which is enough for 140 km. After travelling 55 km, Luca can go a further 85 km. Hence, he cannot reach the 95 km petrol station, but can reach the others. If he stops at the 55 km petrol station (or any nearer one), then he will have at least 410 km left to travel of his 520 km journey, but his tank only holds enough for 400 km. Hence, he should stop at the 75 km petrol station, with 390 km left to travel.

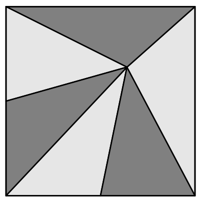
14. The numbers x and y satisfy the equation $17x + 51y = 102$. What is the value of $9x + 27y$?

A 54 B 36 C 34 D 18
E The value is undetermined.

Solution **A**

By dividing $17x + 51y = 102$ by 17 we get $x + 3y = 6$. Multiplying by 9 gives $9x + 27y = 54$.

15. A vertical stained glass square window of area 81 cm^2 is made out of six triangles of equal area (see figure). A fly is sitting on the exact spot where the six triangles meet. How far from the bottom of the window is the fly sitting?



A 3 cm B 5 cm C 5.5 cm D 6 cm E 7.5 cm

Solution **D**

Let h be the height of the fly above the base of the window. Each side-length of the square window of area 81 cm^2 is 9 cm. The two triangles that form the bottom part of the window have total area equal to a third of the whole window, namely 27 cm^2. Hence $\frac{1}{2} \times 9 \times h = 27$ so $h = 27 \times 2 \div 9 = 6$ cm.

16. The digits from 1 to 9 are randomly arranged to make a 9-digit number.

What is the probability that the resulting number is divisible by 18?

A $\frac{1}{3}$ B $\frac{4}{9}$ C $\frac{1}{2}$ D $\frac{5}{9}$ E $\frac{3}{4}$

Solution **B**

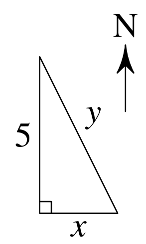
To be divisible by 18, the number must be divisible by 2 and by 9. The digit sum for any number formed is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$, and hence every number is a multiple of 9. Therefore any even number formed will be divisible by 18. Of the 9 possible last digits, there are four even digits (2, 4, 6, 8), so the probability of the number being even is $\frac{4}{9}$.

17. A hare and a tortoise competed in a 5 km race along a straight line, going due North. The hare is five times as fast as the tortoise. The hare mistakenly started running due East. After a while he realised his mistake, then turned and ran straight to the finish point. He arrived at the same time as the tortoise. What was the distance between the hare's turning point and the finish point?

A 11 km B 12 km C 13 km D 14 km E 15 km

Solution **C**

The routes of the hare and the tortoise form a right-angled triangle as shown on the diagram. The hare travels 5 times as fast, but arrives at the same time as the tortoise, so has travelled five times further, giving



$$x + y = 25. \ [1]$$

Pythagoras' Theorem gives $x^2 + 5^2 = y^2$, so $y^2 - x^2 = 25$, and this factorises to give $(y + x)(y - x) = 25$. However, $x + y = 25$ by equation [1]. So $25(y - x) = 25$ and then $y - x = 1$, and $y = x + 1$. Substituting this into equation [1] gives $x + x + 1 = 25$, and hence $x = 12$ and $y = 13$.

18. There are some squares and triangles on the table. Some of them are blue and the rest are red. Some of these shapes are large and the rest are small. We know that

1. If the shape is large, it's a square;
2. If the shape is blue, it's a triangle.

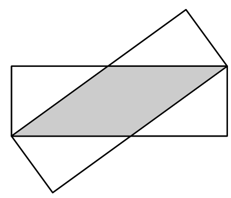
Which of the statements A–E must be true?

A All red figures are squares.
B All squares are large.
C All small figures are blue.
D All triangles are blue.
E All blue figures are small.

Solution **E**

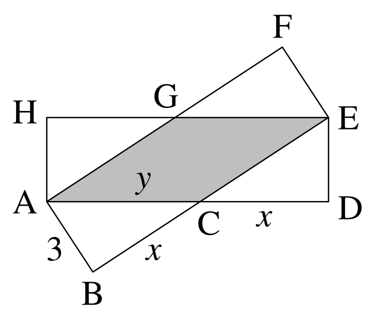
From statement 1 it follows that if a shape is not square, then it is not large; hence all triangles are small. Using statement 2 we see that blue figures are triangles and so are small. This shows that E is true. Suppose we had a set of shapes which consisted of one small blue triangle, one small red triangle, one small red square and one large red square. Then this satisfies statements 1 and 2 but it does not satisfy A, B, C or D. So they are not true in general.

19. Two identical rectangles with sides of length 3 cm and 9 cm are overlapping as in the diagram. What is the area of the overlap of the two rectangles?



A 12 cm^2
B 13.5 cm^2
C 14 cm^2
D 15 cm^2
E 16 cm^2

Solution **D**



First we prove that the four white triangles are congruent. Note that they each have a right angle. Also angles HGA and FGE are equal (vertically opposite). The quadrilateral $ACEG$ is a parallelogram since each of its sides come from the rectangles. Hence angles FGE and GAC are equal (corresponding), and angles GAC and ECD are equal (corresponding). Therefore each triangle has a right-angle, and an angle equal to HGA, and therefore they each have the same angles. Moreover, they each have a corresponding side of length 3 cm. By labelling triangle ABC with lengths 3, x and y, Pythagoras' Theorem gives $x^2 + 3^2 = y^2$ [1]. The triangles ABC and CDE are congruent, so $CD = CB = x$. Since $AD = 9$, we can see $x + y = 9$ [2].

Equation [2] rearranges to $x = 9 - y$, so $x^2 = (9 - y)^2 = 81 - 18y + y^2$. Substituting this into [1] gives $81 - 18y + y^2 + 3^2 = y^2$, so $18y = 90$ and $y = 5$. Thus $x = 9 - 5 = 4$. The area of each white triangle is $\frac{1}{2} \times 3x = \frac{1}{2} \times 3 \times 4 = 6\,\text{cm}^2$. Thus the overlap is $3 \times 9 - 2 \times 6 = 15\,\text{cm}^2$.

20. Kanga labelled the vertices of a square-based pyramid using 1, 2, 3, 4 and 5 once each. For each face Kanga calculated the sum of the numbers on its vertices. Four of these sums equalled 7, 8, 9 and 10. What is the sum for the fifth face?

A 11 B 12 C 13 D 14 E 15

Solution **C**

One face has total 7 which can only be obtained from the given numbers by adding 1, 2 and 4. Therefore it is a triangular face, and so the label of the top vertex, x say, is one of these three values. Hence the square face has 5 at one vertex. So the smallest possible face total for the square is $5 + 1 + 2 + 3 = 11$. Therefore the four face totals given, 7, 8, 9 and 10, must be the face totals of the triangular faces and their sum is 34. Note that each vertex except the top belongs to two triangles; and the top belongs to all four. So the sum 34 is twice the sum of all the labels plus an extra $2x$; that is $34 = 2(1 + 2 + 3 + 4 + 5) + 2x = 30 + 2x$. Hence $x = 2$ and the face total for the square face is $1 + 3 + 4 + 5 = 13$.

21. A large cube is built using 64 smaller identical cubes. Three of the faces of the large cube are painted. What is the maximum possible number of small cubes that can have exactly one face painted?

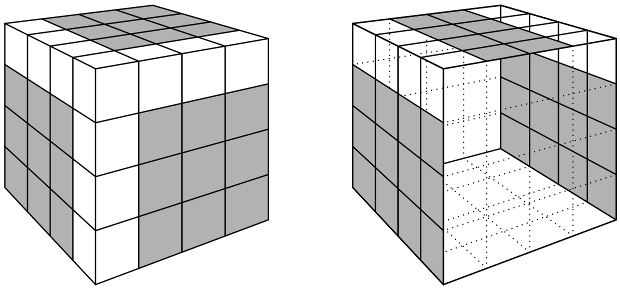
A 27 B 28 C 32 D 34 E 40

Solution **C**

The large cube is $4 \times 4 \times 4$. There are only two possible configurations of the three painted faces: either they all share a common vertex, or they don't.

When the three faces share a common vertex, the edges that they share will have more than one face painted, leaving 9 cubes on each of the three faces with exactly one face painted, as shown in the diagram on the left. This is 27 cubes altogether.

When the three faces don't share a common vertex, then each of the cubes on the common edges will have more than one face painted. This leaves two of the large faces having 12 cubes with one face painted, and the middle large face having 8 cubes, giving a total of 32 cubes, shown in the diagram on the right.



22. In each of the cells, a number is to be written so that the sum of the 4 numbers in each row and in each column are the same.

What number must be written in the shaded cell?

A 5 B 6 C 7 D 8 E 9

1		6	3
	2	2	8
	7		4
(shaded)		7	

Solution **C**

Let x be the number in the bottom right cell. Then the column total is $x + 15$. Since each row and column has the same total, we can now find the other missing values. The third column requires x in its missing cell to make the total up to $x + 15$. The top row requires $x + 5$. The second column requires 1. The bottom row is now missing 7, hence this goes in the shaded square. [The other cells in the left column are $x + 3$ and 4.]

1	$x + 5$	6	3
$x + 3$	2	2	8
4	7	x	4
7 (shaded)	1	7	x

23. Alice, Belle and Cathy had an arm-wrestling contest. In each game two girls wrestled, while the third rested. After each game, the winner played the next game against the girl who had rested. In total, Alice played 10 times, Belle played 15 times and Cathy played 17 times. Who lost the second game?

A Alice
B Belle
C Cathy
D Either Alice or Belle could have lost the second game.
E Either Belle or Cathy could have lost the second game.

Solution **A**

Since each game involved two girls, the number of games played is $(10 + 15 + 17) \div 2 = 21$. Alice played 10 of these, and rested for 11 of them. The maximum amount of resting possible is obtained by alternately losing and resting. To rest 11 times, Alice must have rested the odd-numbered games (1st, 3rd, etc) and lost the even numbered games (2nd, 4th, etc). Hence Alice lost the second game.

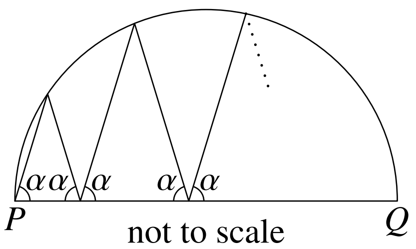
24. Eight consecutive three-digit positive integers have the following property: each of them is divisible by its last digit. What is the sum of the digits of the smallest of these eight integers?

A 9 B 10 C 11 D 12 E 13

Solution **E**

Since we cannot divide by zero, the eight numbers must have the form ABn where n runs from 1 to 8 or from 2 to 9. However if ABn is divisible by n for each n from 2 to 9, it is also divisible by n for each n from 1 to 8 and $AB1$ is smaller than $AB2$. Hence our solution will use the digits 1 to 8. Note that ABn is divisible by n if and only if $AB0$ is divisible by n. So we require the smallest number $AB0$ which is divisible by 1, 2, 3, 4, 5, 6, 7, 8. The LCM of these 8 numbers is $8 \times 7 \times 5 \times 3 = 840$. The eight numbers are then 841, 842, 843, 844, 845, 846, 847, 848. The digit sum of 841 is 13.

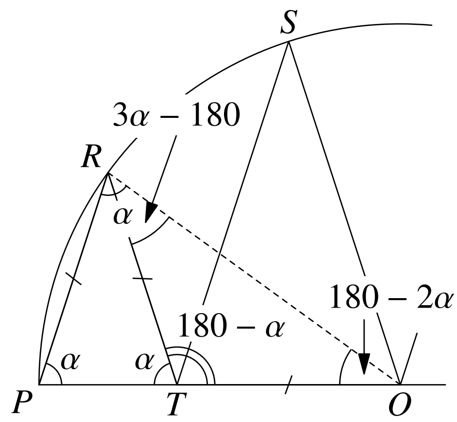
25. A zig-zag line starts at the point P, at one end of the diameter PQ of a circle. Each of the angles between the zig-zag line and the diameter PQ is equal to α as shown. After four peaks, the zig-zag line ends at the point Q. What is the size of angle α?



A 60° B 72° C 75° D 80° E 86°

Solution **B**

After four peaks the zig-zag is at the end of the diameter, so after two peaks it must be at the centre O of the circle. The triangle OPR is isosceles since OP and OR are both radii, hence angle $PRO = \alpha$ and angle $POR = 180° - 2\alpha$. Angle $OTR = 180° - \alpha$ (angles on a straight line), and hence angle $TRO = 3\alpha - 180°$ (angles in triangle TRO).



Triangle OTS is isosceles because its base angles are equal, and hence $ST = SO$. Therefore triangles OPR and OTS are congruent because they both have two sides which equal the radius of the circle, with angle $180° - 2\alpha$ between (SAS). Hence $OT = PR = TR$ (since $PR = RT$ in isosceles triangle PRT). Hence triangle OTR is isosceles, and its base angles are equal, that is $180° - 2\alpha = 3\alpha - 180°$. Solving this gives $\alpha = 72°$.