Solutions to the Olympiad Maclaurin Paper 2015

M1. Consider the sequence 5, 55, 555, 5555, 55 555, ….

Are any of the numbers in this sequence divisible by 495; if so, what is the smallest such number?

Solution

Note that $495 = 5 \times 9 \times 11$, so that a number is divisible by 495 if it is divisible by all of 5, 9 and 11, and not otherwise.

Every number in the sequence has ‘units’ digit 5, so is divisible by 5.

Each even term of the sequence is divisible by 11, but each odd term is not, since it has a remainder of 5 when divided by 11.

It is therefore enough to find the first even term which is divisible by 9.

Suppose a term has $2k$ digits; then its digit sum is $10k$. But a number is divisible by 9 when its digit sum is divisible by 9, and not otherwise. So the first even term divisible by 9 is the one for which $k = 9$.

Therefore the first term divisible by 495 is the 18th term

$$555\,555\,555\,555\,555\,555,$$

which consists of 18 digits 5.

M2. Two real numbers x and y satisfy the equation $x^2 + y^2 + 3xy = 2015$.

What is the maximum possible value of xy ?

Solution

Subtracting x^2 and y^2 from each side of the given equation, we obtain

$$3xy = 2015 - x^2 - y^2.$$

Now adding $2xy$ to each side, we get

$$\begin{aligned} 5xy &= 2015 - x^2 + 2xy - y^2 \\ &= 2015 - (x - y)^2, \end{aligned}$$

which has a maximum value of 2015 when $x = y$.

Therefore xy has a maximum value of 403 and this occurs when $x = y = \sqrt{403}$.

M3. Two integers are *relatively prime* if their highest common factor is 1.

I choose six different integers between 90 and 99 inclusive.

(a) Prove that two of my chosen integers are relatively prime.

(b) Is it also true that two are *not* relatively prime?

Solution

(a) Consider the five pairs (90, 91), (92, 93), (94, 95), (96, 97), and (98, 99); each is a pair of relatively prime integers. Since we are selecting six numbers, we have to choose two from one pair; thus there is a relatively prime pair.

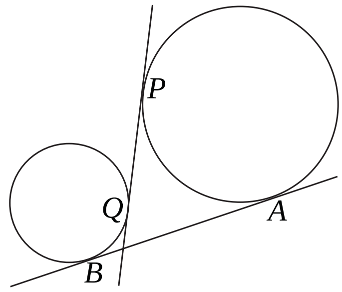
(b) If two or more of the six numbers chosen are even, then they have a common factor of 2.

If only one is even, then the others are 91, 93, 95, 97 and 99. Therefore two of the chosen numbers (93 and 99) have a common factor of 3.

In either case, two of my six chosen numbers have a common factor greater than 1, so are not relatively prime.

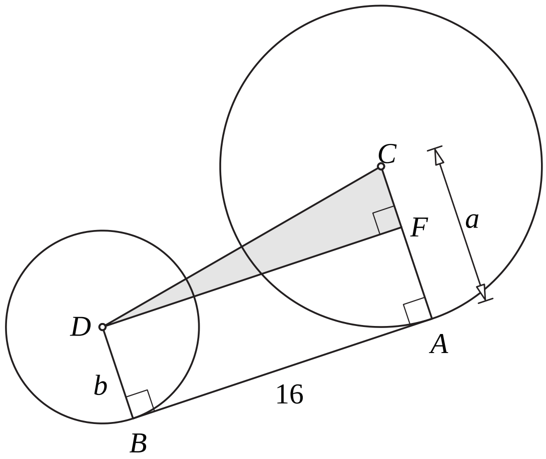
M4. The diagram shows two circles with radii a and b and two common tangents AB and PQ. The length of PQ is 14 and the length of AB is 16.

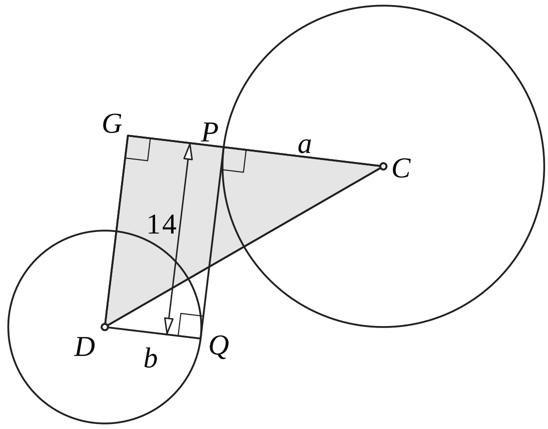
Prove that $ab = 15$.



Solution

Let the centres of the circles be C and D.





Firstly, consider the tangent AB (see the left-hand figure above). Join C to A and D to B, forming two right angles, as shown, since tangent and radius are perpendicular. Let F be the foot of the perpendicular from D to CA. Then $AFDB$ is a rectangle, so that $AF = b$ and $FD = 16$.

From Pythagoras' Theorem for right-angled triangle CDF, we therefore obtain

$$CD^2 = 16^2 + (a - b)^2. \tag{1}$$

Next, consider the tangent PQ (see the right-hand figure above). Join C to P and D to Q, once again forming two right angles, as shown, since tangent and radius are perpendicular. Let G be the foot of the perpendicular from D to CP. Then $PGDQ$ is a rectangle, so that $PG = b$ and $GD = 14$.

From Pythagoras' Theorem for right-angled triangle DCG, we therefore obtain

$$CD^2 = 14^2 + (a + b)^2. \tag{2}$$

It follows from equations (2) and (1) that

$$14^2 + (a + b)^2 = 16^2 + (a - b)^2,$$

and hence

$$(a + b)^2 - (a - b)^2 = 16^2 - 14^2.$$

In order to simplify this equation we could just multiply out the brackets, but it is a little quicker to use the ‘difference of two squares’ factorisation $x^2 - y^2 = (x - y)(x + y)$ for each side, which also avoids any squaring! On doing this, we get

$$2b \times 2a = 2 \times 30$$

and hence

$$ab = 15.$$

Now in the diagrams we assigned each of a and b to be the radius of a particular circle, whereas the question does not do so. However, interchanging a and b in the result leaves it unchanged, which means that the result is independent of the assignment of letters.

M5. Consider equations of the form $ax^2 + bx + c = 0$, where a, b, c are all single-digit prime numbers.

How many of these equations have at least one solution for x that is an integer?

Solution

Note that a, b and c, being prime, are positive integers greater than 1.

If the equation has an integer solution, then the quadratic expression factorises into two 'linear' brackets (...)(...).

Since c is prime, the 'constant' terms in the brackets can only be c and 1. Similarly, since a is prime, the x terms can only be ax and x. Hence the factorisation is either $(ax + c)(x + 1)$, in which case $b = a + c$, or $(ax + 1)(x + c)$, in which case $b = ac + 1$. Each case leads to an integer solution for x, either $x = -1$ or $x = -c$.

In the first case exactly one of a or c is 2, since b is prime and $b = a + c$. Therefore (a, b, c) is one of (2, 5, 3), (2, 7, 5), (3, 5, 2) or (5, 7, 2).

In the second case one or both of a and c is 2, since b is prime and $b = ac + 1$. Therefore (a, b, c) is one of (2, 5, 2), (2, 7, 3) or (3, 7, 2).

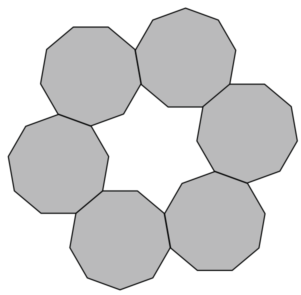
Hence altogether there are seven such quadratic equations.

M6. A symmetrical ring of m identical regular n-sided polygons is formed according to the rules:

(i) each polygon in the ring meets exactly two others;

(ii) two adjacent polygons have only an edge in common; and

(iii) the perimeter of the inner region–enclosed by the ring–consists of exactly two edges of each polygon.

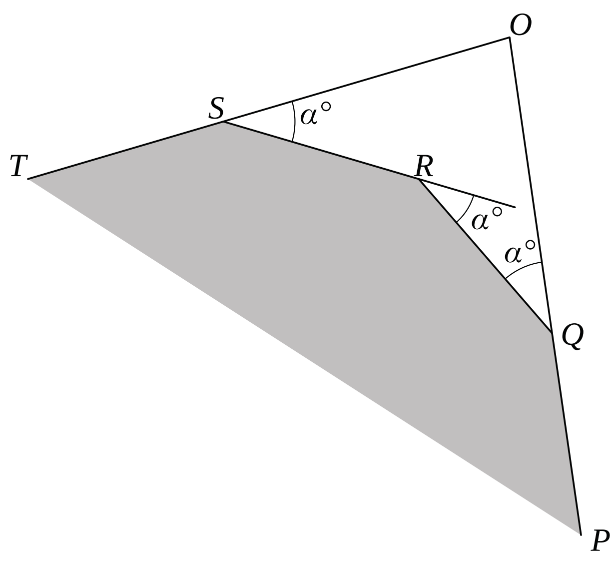


The example in the figure shows a ring with $m = 6$ and $n = 9$.

For how many different values of n is such a ring possible?

Solution

Let the exterior angle in each polygon be $\alpha°$, so that $\alpha = \frac{360}{n}$, and let P, Q, R, S and T be five consecutive vertices of one of the polygons, as shown, where QRS is part of the perimeter of the inner region.



From the symmetry, the point O where PQ and TS meet is the centre of the inner region and hence

$$\angle QOS = \frac{360°}{m}.$$

Now the sum of the angles in quadrilateral $OSRQ$ is 360°, so we have

$$360 = \frac{360}{m} + \alpha + (180 + \alpha) + \alpha$$

and therefore

$$1 = \frac{2}{m} + \frac{6}{n}.$$

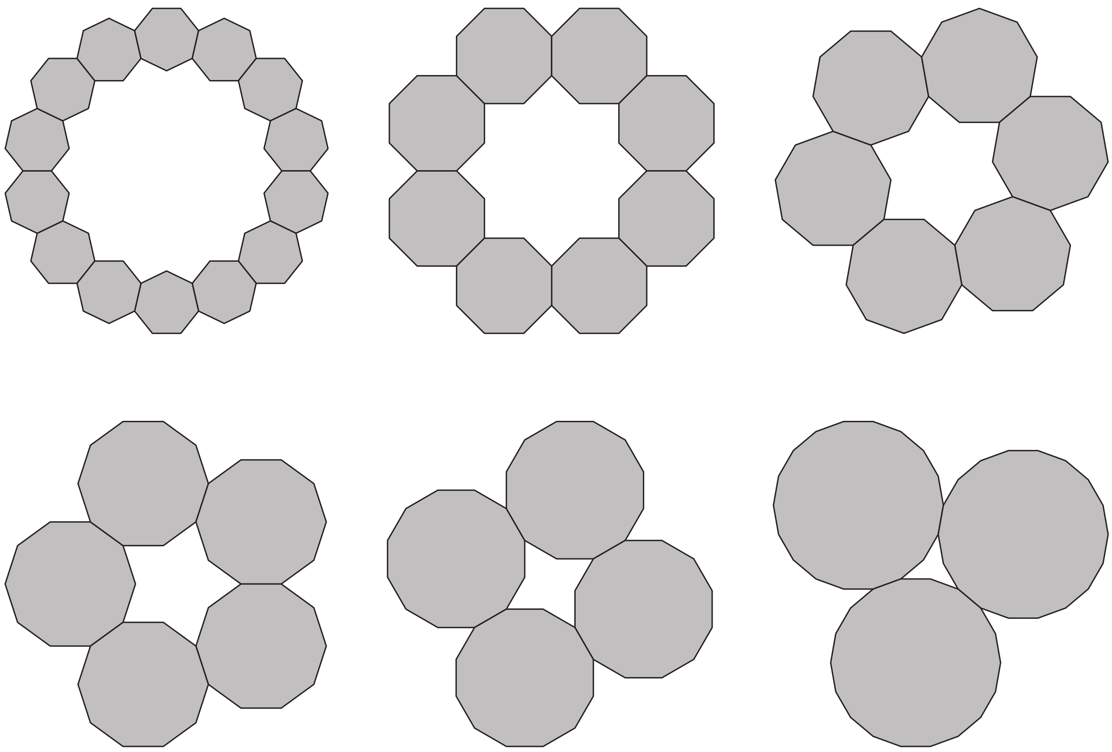
Multiplying each term by mn, we obtain

$$mn = 6m + 2n,$$

so that

$$(m - 2)(n - 6) = 12.$$

Now $m > 2$ for the ring to exist. Hence $(m - 2)(n - 6)$ is a product of two positive integers, so that $n - 6$ is a positive factor of 12. The only possibilities for $n - 6$ are therefore 1, 2, 3, 4, 6 and 12, and thus the only possibilities for n are 7, 8, 9, 10, 12 and 18. The following figures show the corresponding rings.



Thus a ring of polygons of the required form is possible for six different values of n.