Junior Mathematical Olympiad 2011

The Junior Mathematical Olympiad (JMO) has long aimed to help introduce able students to (and to encourage in) the art of problem-solving and proof. The problems are the product of the imaginations of a small number of volunteers writing for the JMO problems group. After the JMO, model solutions for each problem are published in the solutions booklet and the UKMT Yearbook.

However, these neatly printed solutions usually represent only one approach (often where several are feasible) and convey none of the sense of investigation, rough-working and false-starts that usually precede the finished article.

In marking JMO scripts, markers want to encourage complete (and concise) proofs in section B but, in doing so, are content to accept less than perfectly written answers, as long as it is clear where the proof is going – we appreciate that there is a limited time available, and that to polish and to present takes the place of considering another problem. Nevertheless, clarity and insight are what are looked for.

We have collected below a small number of solutions submitted for the JMO in 2011, with the aim that future candidates can see what some Year 8 students (and younger ones) do achieve and that they might aspire to emulate it. In many ways they are ordinary solutions, not brilliantly and startlingly original – but mathematically they are to be commended, in particular, for the logical progression from one point to the next, and for clear presentation.

It is not easy to generalise about what makes a good solution or how a candidate can achieve success in the JMO, but there are a few points to ponder on:

- using trial and improvement to a large degree is generally considered not very mathematical – it will lend little insight into the structure of the problem, and even if you get an answer that works, there should always be a concern that it is not the only answer;
- if you do have to resort to calculating your way through a large number of cases, then the calculations should be shown as part the solution – the reader should not have to make suppositions about what you have tried (or not tried);
- diagrams should be large enough to contain subsequent details – however, drawing just one diagram can convey very little of the order of the proof, especially in geometrical questions;
- avoid long sentences, particularly those which frequently use *if* and *could be*.
- the use of algebra makes it possible to express connections in simple ways, where a huge number of words would be otherwise necessary;
- if algebra is used, variables should be fully declared, as in B2, B4 and B5 below;
- even though simplifying can often lead to insight into a more general problem, you should take care not to oversimplify solution, and this was particularly apparent in question B5, where it was generally wrongly assumed (perhaps, without noticing) that B could be placed exactly halfway between A and C (after which the solution became rather trivial) – in the solution for B5 below, the candidate has taken care to refer to the distances AB and BC using different letters, leading to the more general solution that was intended;
- attempt as many questions as you can do really well in – it is better, in terms not only of scoring marks but also of honest satisfaction, to spend your time concentrating on a few questions and providing full, clear and accurate solutions, rather than to have a go at everything you can, and to achieve not very much in any of them. Of the 1000 students or so who take part in the JMO, generally only a handful achieve full marks in all six section B questions.

If this all sounds rather negative, it has, in balance, to be said that JMO candidates produce an astonishing amount of work that is worthy of high praise. We hope that the work below will give future candidates a flavour of what is to be encouraged, in style, method and presentation.

B1 Every digit of a given positive integer is either a 3 or a 4 with each occurring at least once. The integer is divisible by both 3 and 4.

What is the smallest such integer?

Solution:

The number cannot have 1 digit as there must be a digit of 3 and a digit of 4. If the number had 2 digits it could be either 34 or 43, but neither of these is divisible by 12. (As 12 is the lcm (lowest common multiple) of 3 and 4, any number divisible by 12 is also divisible by both 3 and 4). If the integer had 3 digits, the lowest number it could be is 334, and then 343, 344, 433, 434 or 443. As none of the digits of these numbers add up to a number divisible by 3, none of these divide by 3. If the integer had 4 digits it could be one of: 3334, 3343, 3433, 4333, 3344, 3434, 4334, 3443, 4343, 4433, 3444, 4344, 4434 or 4443. The digits of the last 4 numbers <u>only</u> add up to 15 so only ~~theirr~~ are divisible by 3. As the smallest of those, 3444, is divisible by 12 (3444 ÷ 12 = 287), the smallest such number is 3444.

B2 A 3 × 3 grid contains nine numbers, not necessarily integers, one in each cell. Each number is doubled to obtain the number on its immediate right and trebled to obtain the number on its immediate right and trebled to obtain the number immediately below it.

If the sum of the nine numbers is 13, what is the value of the number in the central cell?

Solution:

If we call the number in the top left corner x, then we can say the one to the right of it is $2x$ and the one below it is $3x$. If we work this out for each cell, we get the following.

x	$2x$	$4x$
$3x$	$6x$	$12x$
$9x$	$18x$	$36x$

Adding these together, we find the total of the nine cells is $91x$. We already know the cells add up to 13,

so: $91x = 13$

$$x = \frac{13}{91}$$

$$x = \frac{1}{7}$$

If $x = \frac{1}{7}$ and the central cell $= 6x$, then the number in the central cell must be $\frac{6}{7}$ because it's $\frac{1}{7} \times 6$.

B3 When Dad gave out the pocket money, Amy received twice as much as her first brother, three times as much as the second, four times as much as the third and five times as much as the last brother. Peter complained that he had received 30p less than Tom.

Use this information to find all the possible amounts that Amy could have received.

Solution:

Since the lcm of 1,2,3,4 & 5 is 60 then the lowest amount of money Amy could have received is 60p. In this case she and her siblings would get 60p, 30p, 20p, 15p and 12p. Since Tom and Peter could have been any combination of brothers from 1-4 (of which there are 6 combinations (1-2, 1-3, 1-4, 2-3, 2-4, 3-4)) then there are 6 different amounts of money Amy could have received. (or are there?)

· If Tom & Peter are brothers 1 & 2 (30p & 20p) they must have got 90p and 60p, so Amy would get £1·80.

· If they are brothers 1 & 3 (30p and 15p) they will have got 60p and 30p, so Amy would get £1·20

· If they were 1 & 4 (30p and 12p) they would have got 50p and 20p. However, this would mean that Amy would get £1 and brother 2 would get $33\frac{1}{3}$p, so this doesn't work!

If they were brothers 2 & 3 (20p and 15p) they would have got £1·20 and 90p so Amy would get £2·40

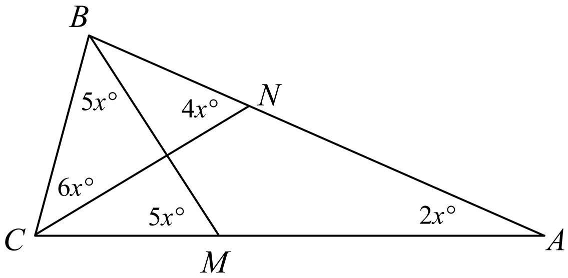
→ If they were 2 & 4 (20p and 12p) they would get 75p and 45p and Amy would get £2·25. However, brother 1 would get £1·12·5 so this doesn't work!

→ Finally, if they were 3 & 4 (15p and 12p) they will have got £1·50 and £1·20 so Amy would get £6.

The totals are: £1·80, £1·20, £2·40 and £6·00

B4 In a triangle ABC, M lies on AC and N lies on AB so that $\angle BAC = 2x°$, $\angle BNC = 4x°$, $\angle BMC = \angle CBM = 5x°$ and $\angle BCN = 6x°$.

Prove that triangle ABC is isosceles.



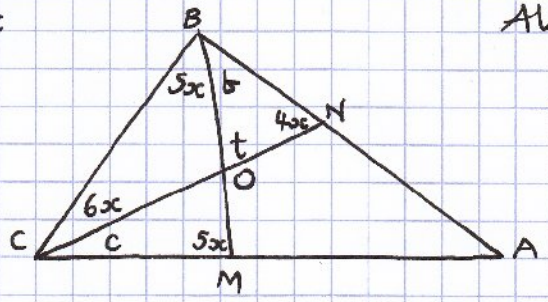
Solution:

In order for a triangle to be isosceles, two of the angles must be equal.

I'm going to call where BM and CN cross O.

And angle MBN = $b°$ and angle NCM = $c°$.

So we've got:



Also let angle $\angle BON = t°$

In a triangle, three angles add up to 180°

So in triangle BON $\quad b + t + 4x = 180 \quad (1)$

The angle COM is vertically opposite to BON, and so is $t°$

Now in triangle COM $\quad c + t + 5x = 180 \quad (2)$

Using equations (1) and (2)

$$b + t + 4x = c + t + 5x$$

so $$b + 4x = c + 5x$$

so $$b = c + x$$

But this means that angle $ABC = 5x + b$
$= 6x + c$

which is exactly the same as angle ACB.

Therefore angles ABC and ACB are equal.

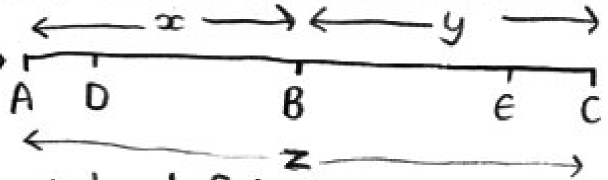
So triangle ABC is isosceles.

B5 Calum cycles from A to C, passing through B.
During the trip he asks his friend how far they have cycled.
His friend replies “one third as far as it is from here to B”.
Ten miles later Calum asks him how far it is to C.
His friend replies again “one third as far as it is from here to B”.

How far from A will Calum have cycled when he reaches C?

Solution:

Look at this diagram →



D is where question 1 was asked and E is where question 2 was asked. Since $DB = 3AD$, $AD = \frac{x}{4}$. Likewise, $EC = \frac{y}{4}$. Since $DE = 10$, we can get: $z = \frac{x}{4} + \frac{y}{4} + 10$

$\rightarrow 4z = 4(x/4 + y/4 + 10)$

$\rightarrow 4z = x + y + 40$

Since $x + y = z$, we get:

$4z = 40 + z$

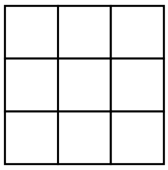
$\rightarrow 3z = 40$

$\rightarrow z = 13\frac{1}{3}$

so the distance from A to C is $13\frac{1}{3}$ miles.

B6 Pat places counters in some of the cells of the 3 × 3 grid shown, then finds the total number of counters in each row and each column.
Pat is trying to place counters in such a way that these six totals are all different.

What is the smallest number of counters that Pat can use?



Solution:

We need to find the smallest number of counters that Pat can use. Let it be x

0	1	2	3
3	4	7	14
5	6	8	19
8	11	17	

Row totals: $3 + 14 + 19 = 36$

Column totals: $8 + 11 + 17 = 36$

From the above diagram I concluded that the total of all the of the column totals is always the same as the total of all the row totals

This means that if we add the 3 column totals to the 3 row totals we get twice the number of counters ($2x$)

Pat is trying to make all the totals different

$\therefore$ the smallest total is $0+1+2+3+4+5 = 15$

Because x is an integer the smallest value of x is 8
As $2x \geqslant 15$

By experimentation I found this solution

			0
	2		2
1	1	4	6
1	3	4	

Which uses 8 counters

So the smallest number she can use 8