UK JUNIOR MATHEMATICAL CHALLENGE

April 26th 2012

SOLUTIONS

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some ***Extension Problems*** for further investigations.

The Junior Mathematical Challenge (JMC) is a multiple choice contest, in which you are presented with five alternative answers, of which just one is correct. Sometimes you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. Also, no reasons for the answers need to be given. However, providing good and clear explanations is the heart of doing Mathematics. So here we have aimed to give full solutions with all steps explained. We therefore hope that these solutions can be used as a model for the type of written solution that is expected in the Junior Mathematical Olympiad and similar competitions.

We welcome comments on these solutions, and, especially, corrections or suggestions for improving them. Please send your comments,

either by e-mail to enquiry@ukmt.co.uk, or by post to JMC Solutions, UKMT Maths Challenges Office, School of Mathematics, University of Leeds, Leeds LS2 9JT.

Quick Marking Guide

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
E	C	E	C	D	D	B	B	C	E	B	B	C	A	E	D	A	D	E	A	B	B	A	D	A

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1. What is the smallest four-digit positive integer which has four different digits?

A 1032 B 2012 C 1021 D 1234 E 1023

Solution: **E**

Here it is easy just to check the options that are given. A, D and E are the only options in which all four digits are different. Of these, clearly, E is the smallest.

For a complete solution we need to give an argument to show that 1023 really is the smallest four-digit positive integer with four different digits. It is easy to do this.

To get the smallest possible number we must use the four smallest digits, 0, 1, 2 and 3. A four digit number cannot begin with a 0. So we must put the next smallest digit, 1, in the thousands place, as a four-digit number beginning with 2 or 3 is larger than one beginning with a 1. For similar reason the hundreds digit must be the smallest remaining digit, 0. Similarly the tens digit must be 2 and the units digit must be 3. So the required number is 1023.

Extension problem

1.1 What is the smallest ten-digit positive integer which has ten different digits?

2. What is half of 1.01?

A 5.5 B 0.55 C 0.505 D 0.5005 E 0.055

Solution: **C**

We obtain half of 1.01 by dividing 1.01 by 2. We can do this as a long division sum:

$$2\overline{)1.010}\quad = 0.505 .$$

Alternatively, we can use fractions:

$1.01 = 1 + \frac{1}{100}$. So half of 1.01 is $\frac{1}{2} + \frac{1}{200} = 0.5 + 0.005 = 0.505$.

3. Which of the following has exactly one factor other than 1 and itself?

A 6 B 8 C 13 D 19 E 25

Solution: **E**

The factors of 6 are 1, 2, 3 and 6; the factors of 8 are 1, 2, 4 and 8; the factors of 13 are 1, 13; the factors of 19 are 1, 19; and the factors of 25 are 1, 5, 25. We see from this that, of the numbers we are given as options, only 25 has exactly one factor other than 1 and itself.

Extension problems

3.1 Find all the positive integers less than 25 which have exactly one factor other than 1 and itself.

3.2 Show that a positive integer has exactly one factor other than 1 and itself if and only if it is the square of a prime number.

4. Beatrix looks at the word **JUNIOR** in a mirror.

How many of the reflected letters never look the same as the original, no matter how Beatrix holds the mirror?

A 1 B 2 C 3 D 4 E 5

Solution: **C**

The letters **J**, **N** and **R** do not have an axis of symmetry. So these letters cannot look the same when reflected in a mirror, however the mirror is held. The letters **U**, **I** and **O** all have at least one axis of symmetry. So each may look the same when reflected in a mirror.

Extension problem

4.1 Count the number of symmetries (reflections and rotations) of each letter of the alphabet. (The answer will depend on the font you use. For example, **K** does not have an axis of symmetry, but **K** does.)

5. One of the mascots for the 2012 Olympic Games is called 'Wenlock' because the town of Wenlock in Shropshire first held the Wenlock Olympian Games in 1850. How many years ago was that?

A 62 B 152 C 158 D 162 E 172

Solution: **D**

All you have to do is a subtraction: $2012 - 1850 = 162$.

For more information about the Wenlock Olympian Games, go to their website:

http://www.wenlock-olympian-society.org.uk/

6. The diagrams on the right show three different views of the same cube. Which letter is on the face opposite U?

A I B P C K D M E O

Solution: **D**

From the left hand diagram we see that K is not opposite either I or M, and from the middle diagram that it is not opposite either O or U. Therefore K must be opposite P. So neither K nor P is opposite U. From the middle diagram O is also not opposite U. So U is opposite either I or M. But if U is opposite I, then O must be opposite M, and this possibility is ruled out by the right hand diagram. So U must be opposite to M (and, also, I is opposite O).

7. A small ink cartridge has enough ink to print 600 pages. Three small cartridges can print as many pages as two medium cartridges. Three medium cartridges can print has many pages as two large cartridges. How many pages can be printed using a large cartridge?

A 1200 B 1350 C 1800 D 2400 E 5400

Solution: **B**

Three small cartridges have enough ink for $3\times 600=1800$ pages. So 1800 is the number of pages that two medium cartridges can print. Hence one medium cartridge can print $\frac{1}{2}(1800)=900$ pages. So three medium cartridges have enough ink for $3\times 900=2700$ pages. So 2700 is the number of pages that two large cartridges can print. Hence one large cartridge can print $\frac{1}{2}(2700)=1350$ pages.

An alternative, algebraic method, is to let *m* be the number of pages that a medium cartridge can print, and *l* be the number of pages that a large cartridge can print. From the information we are given we have the equations

$$3\times 600=2m \text{ and } 3m=2l .$$

From these we can deduce that

$$l=\frac{3}{2}m=\frac{3}{2}\times\frac{3\times 600}{2}=9\times 150=1350 .$$

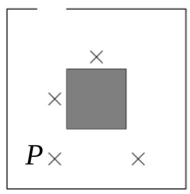
8. Tommy Thomas's tankard holds 480 ml when it is one quarter empty. How much does it hold when it is one quarter full?

A 120 ml B 160 ml C 240 ml D 960 ml E 1440 ml

Solution: **B**

When Tommy's tankard is one quarter empty it is three quarters full. So 480 ml is three quarters of the capacity of the tankard. So when it is one quarter full it holds $\frac{1}{3}(480)=160$ ml.

9. The diagram on the right shows the positions of four people (each marked ×) in an Art Gallery. In the middle of the room is a stone column. Ali can see none of the other three people. Bea can see only Caz. Caz can see Bea and Dan. Dan can only see Caz.

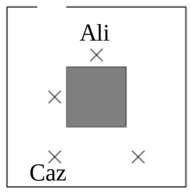


Who is at position *P*?

A Ali B Bea C Caz D Dan E More information needed.

Solution: **C**

The person in position *P* is the only one who can see two of the others. We are told that Caz can see Bea and Dan but everyone else can see just one other person or no-one. So it must be Caz who is at position *P*.

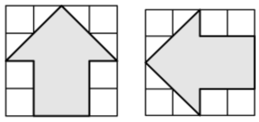


(We can also see that Ali, who can see no-one, must be in the position shown. Bea and Dan must be in the other two positions, but the information we are given doesn't enable us to work out which is where.)

10. The diagram shows two arrows drawn on separate 4 cm × 4 cm grids.

One arrow points North and the other points West.

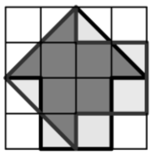
When the two arrows are drawn on the same 4 cm × 4 cm grid (still pointing North and West) they overlap. What is the area of overlap?



A 4 cm^2 B $4\frac{1}{2}\text{ cm}^2$ C 5 cm^2 D $5\frac{1}{2}\text{ cm}^2$ E 6 cm^2

Solution: **E**

By drawing one arrow on top of the other, as shown, we see that the region of overlap covers the whole of 4 of the 1 cm × 1 cm squares into which the grid is divided, and 4 halves of these squares. So the area of the overlapping region is $4 + 4(\frac{1}{2}) = 6\text{ cm}^2$.



11. In the following expression each ☐ is to be replaced with either + or − in such a way that the result of the calculation is 100.

123 ☐ 45 ☐ 67 ☐ 89

The number of + signs used is p and the number of − signs used is m. What is the value of $p - m$?

A -3 B -1 C 0 D 1 E 3

Solution: **B**

The sum is made up of 123 and ± 45, ± 67 and ± 89. Suppose that the total of the positive terms in the calculation is x and the total of the negative terms is y. So $x > 0$ and $y < 0$. We need to have that

$$x + y = 100.$$

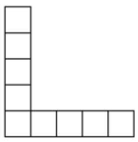
We also have that

$$x - y = 123 + 45 + 67 + 89 = 324.$$

Adding these equations, we obtain $2x = 424$. So $x = 212$ and hence $y = -112$. It is readily seen that $45 + 67 = 112$ and that no other combination of 45, 67 and 89 gives a total of 112. So the correct calculation must be $123 - 45 - 67 + 89 = 100$ with 1 plus sign and 2 minus signs. So $p = 1$ and $m = 2$, giving $p - m = -1$.

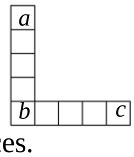
12. Laura wishes to cut this shape, which is made up of nine small squares, into pieces that she can then rearrange to make a 3 × 3 square.

What is the smallest number of pieces that she needs to cut the shape into so that she can do this?

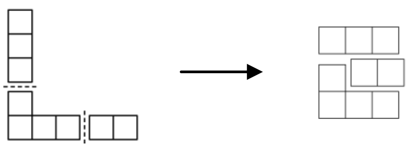


A 2 B 3 C 4 D 5 E 6

Solution: **B**



In a 3×3 square each row and column contains just 3 squares. So none of the pieces that Laura uses to make the square can be more than 3 squares long. It follows that the squares labelled a and b must be in different pieces, as also must be the squares labelled b and c. So there must be at least three pieces. The diagrams below show how Laura can fulfill her task using 3 pieces.



13. In the multiplication grid on the right, the input factors (in the first row and the first column) are all missing and only some of the products within the table have been given.

What is the value of A + B + C + D + E ?

A 132 B 145 C 161 D 178 E 193

×					
	A	10		20	
	15	B	40		
	18		C	60	
		20		D	24
			56		E

Solution: **C**

×	p	q	r	s	t
v	A	10		20	
w	15	B	40		
x	18		C	60	
y		20		D	24
z			56		E

Let the input factors be p, q, r, s and t along the top row, and v, w, x, y and z in the first column, as shown in the grid on the right.

We see that $wp = 15$ and $xp = 18$, so that p is a divisor of both 15 and 18, and hence is either 1 or 3. However, if $p = 1$, then it would follow that $w = 15$. But this is not possible as $wr = 40$. We deduce that $p = 3$. It follows that $w = 5$ and $x = 6$. Since $w = 5$ and $wr = 40$, we have that $r = 8$. Therefore, as $zr = 56$, $z = 7$. Also, as $x = 6$ and $xs = 60$, $s = 10$. Since $vs = 20$, it follows that $v = 2$. Then, as $vq = 10$, $q = 5$. Hence, as $yq = 20$, we have that $y = 4$. Finally, as $yt = 24$, we deduce that $t = 6$.

×	3	5	8	10	6
2	**6**	10	16	20	12
5	15	**25**	40	50	30
6	18	30	**48**	60	36
4	12	20	32	**40**	24
7	21	35	56	70	**42**

This enables us to complete the table, as shown on the left. (Though, really, we need only to calculate the diagonal entries that we have shown in bold.)

Therefore, we have $A + B + C + D + E = 6 + 25 + 48 + 40 + 42 = 161$.

14. A pattern that repeats every six symbols starts as shown below:

♥ ♣ ♥ ♦ ♥ ♠ ♥ ♣ ♥ ♦ ♥ ♠ . . .

Which are the 100th and 101st symbols, in that order, in the pattern?

A ♦ ♥ B ♥ ♦ C ♥ ♣ D ♠ ♥ E ♣ ♥

Solution: **A**

The pattern repeats every 6 symbols. Therefore, as 96 is a multiple of 6, the symbols in positions 97, 98, 99, 100, 101, 102, are the same as those in positions 1, 2, 3, 4, 5, 6, that is, they are ♥ ♣ ♥ ♦ ♥ ♠ . We see from this that the symbols that are the 100th and 101st in the list are ♦ ♥, in this order.

Extension problems

14.1 Which is the 1001st symbol?

14.2 Suppose that p is a prime number. What are the possibilities for the pth symbol?

15. Talulah plants 60 tulip bulbs. When they flower, she notes that half are yellow; one third of those which are not yellow are red; and one quarter of those which are neither yellow nor red are pink. The remainder are white. What fraction of the tulips are white?

A $\frac{1}{24}$ B $\frac{1}{12}$ C $\frac{1}{6}$ D $\frac{1}{5}$ E $\frac{1}{4}$

Solution: **E**

Since half the 60 bulbs are yellow, 30 are yellow and 30 are not yellow. One third of the 30 bulbs that are not yellow are red, So 10 are red and 20 are neither yellow nor red. One quarter of the 20 that are neither yellow nor red, are pink. So 5 bulbs are pink. This leaves 15 bulbs which are neither yellow, nor red nor pink. So there are 15 white bulbs. Therefore the fraction of bulbs that are white is $\frac{15}{60}$, that is, $\frac{1}{4}$.

16. Beth, Carolyn and George love reading their favourite bedtime stories together. They take it in turns to read a page, always in the order Beth, then Carolyn, then George. All twenty pages of the story are read on each occasion. One evening, Beth is staying at Grandma's house but Carolyn and George still read the same bedtime story and take it in turns to read a page with Carolyn reading the first page.

In total, how many pages are read by the person who usually reads that page?

A 1 B 2 C 4 D 6 E 7

Solution: **D**

When all three children are present, Carolyn reads pages 2, 5, 8, … , that is those pages whose number leaves remainder 2 when divided by 3. George reads pages 3, 6, 9, ..that is, those whose number is a multiple of 3. When Beth stays at Grandma's, Carolyn reads pages 1, 3, 5, .. that is the odd numbered pages, and George reads the even numbered pages. So the pages that Carolyn reads both normally and also when Beth is away are those with numbers which have remainder 2 when divided by 3, and are odd. That is, the three pages 5, 11 and 17. The pages that George reads both normally and also when Beth is away are those with numbers that are multiples of 3 and are even, that is, the three pages 6, 12 and 18. So there are altogether 6 pages which are read by the same person normally and when Beth is away, namely pages 5, 6, 11, 12, 17 and 18.

Extension problem

16.1 One day Beth, Carolyn and George read a book of 240 pages. They take it in turn to read a page, always in the order Beth, Carolyn and George. The next day their cousin Sam comes to stay and they read the book again, taking it turns to read a page, always in the order Sam, George, Carolyn and Beth. How many pages are read by the same person on the two days?

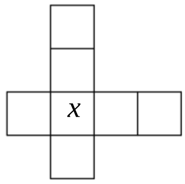
17. There are six more girls than boys in Miss Spelling's class of 24 pupils. What is the ratio of girls to boys in this class?

A 5:3 B 4:1 C 3:1 D 1:4 E 3:5

Solution: **A**

Suppose there are *g* girls and *b* boys in Miss Spelling's class. As there are 6 more girls than boys, $g-b=6$. As there are 24 pupils altogether, $g+b=24$. Adding these equations gives $2g=30$, so $g=15$. Hence $b=15-6=9$. So the ratio *g*:*b* is 15:9 which is the same as 5:3.

18.	The numbers 2, 3, 4, 5, 6, 7, 8 are to be placed, one per square, in the diagram shown such that the four numbers in the horizontal row add up to 21 and the four numbers in the vertical column add up to 21. Which number should replace x? A 2 B 3 C 5 D 7 E 8



Solution: **D**

If we add all the numbers in the horizontal column and all the numbers in the vertical row, we get a total of $21+21=42$. In doing this sum we add in all the numbers 2, 3, 4, 5, 6, 7, 8 once except for x which is added in twice. So the total we get is $2+3+4+5+6+7+8+x=35+x$. Since this equals 42, we must have $x=7$.

To complete the solution we should check that with $x=7$, it is possible to place the remaining numbers in the other squares so that the four numbers in the horizontal row add up to 21, and so also do the four numbers in the vertical column. We ask you to do this in Extension Problem 18.1

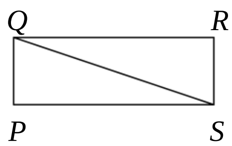
Extension problem

18.1 Show that with $x=7$, it is possible to place the remaining numbers, 2, 3, 4, 5, 6 and 8, in the other squares so that the four numbers in the horizontal row add up to 21, and so also do the four numbers in the vertical column. We ask you to do this in Extension Problem 18.1

18.2 In how many different ways can the numbers 2, 3, 4, 5, 6, 7, 8 be placed in the squares so that the four numbers in the horizontal row add up to 21, and the four numbers in the vertical column also add up to 21?

19.	In rectangle *PQRS*, the ratio of $\angle PSQ$ to $\angle PQS$ is 1:5. What is the size of $\angle QSR$? A 15° B 18° C 45° D 72° E 75°

Solution: **E**

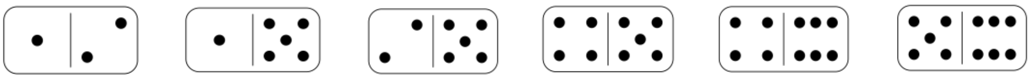


Let $\angle QSR = x^\circ$. Since *PS* is parallel to *QR*, the alternate angles $\angle PQS$ and $\angle QSR$ are equal. So $\angle PQS = x^\circ$. Therefore, as $\angle PSQ : \angle PQS$ $=1:5$, $\angle PSQ = \frac{1}{5}x^\circ$. Therefore, from the right angled triangle *PQS* we deduce that $x + \frac{1}{5}x + 90 = 180$ and so $\frac{6}{5}x = 90$. Therefore $x = \frac{5}{6} \times 90 = 75$. So $\angle QSR$ is 75°.

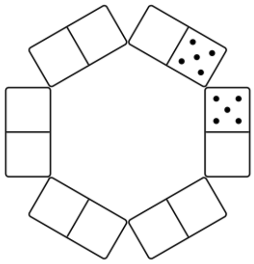
20.	Aaron says his age is 50 years, 50 months, 50 weeks and 50 days old. What age will he be on his next birthday? A 56 B 55 C 54 D 53 E 51

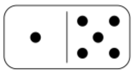
Solution: **A**

50 months is 4 years and 2 months. 50 weeks is around $11\frac{1}{2}$ months and 50 days is about $1\frac{2}{3}$ months. So Aaron's age is approximately $(50+4)$ years $+(2+11\frac{1}{2}+1\frac{2}{3})$ months $=54$ years $+ 15\frac{1}{6}$ months $=55$ years and $3\frac{1}{6}$ months. So he will be 56 on his next birthday.

21. 

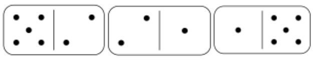
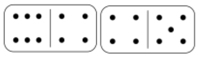
Dominic wants to place the six dominoes above in a hexagonal ring so that, for every pair of adjacent dominoes, the numbers of pips math. The ring on the right indicates how one adjacent pair match.

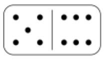
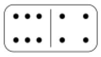


In a completed ring, how many of the other five dominoes can he definitely *not* place adjacent to  ?

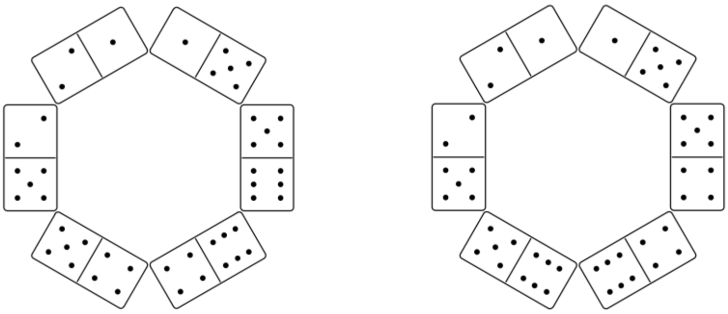
A 1 B 2 C 3 D 4 E 5

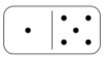
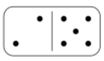
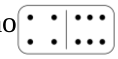
Solution: **B**

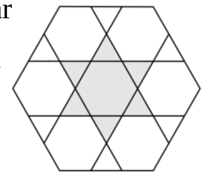
There are only two 1-pip dominoes among the five Dominic has. These must therefore be adjacent. Likewise for the two 2-pips and the two 4-pips. So the ring must include the three adjacent dominoes  and also the two adjacent dominoes  . The ring

may now be completed by placing the remaining domino  adjacent to the 5-pips at either end of the block of three dominoes, and to the 6-pips on the domino  . In either case this leaves two 5-pips dominoes which can then be placed adjacent to each other to complete the ring.

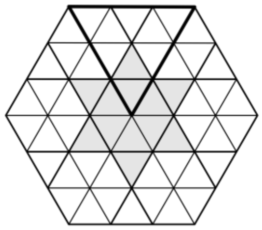
We therefore see that Dominic can create two different rings of six dominoes, as shown below.



We now see that there are just 2 dominoes that the domino  cannot be adjacent to, namely the domino  and, of course, the domino  .

22. The diagram shows a design formed by drawing six lines in a regular hexagon. The lines divide each edge of the hexagon into three equal parts. What fraction of the hexagon is shaded?  A $\frac{1}{5}$ B $\frac{2}{9}$ C $\frac{1}{4}$ D $\frac{3}{10}$ E $\frac{5}{16}$

Solution: **B**



If we draw in the additional lines shown in the diagram on the right, the hexagon is divided into 54 small congruent equilateral triangles of which 12 are shaded. So the fraction that is shaded is $\frac{12}{54} = \frac{2}{9}$.

[In fact, it is easier to note that the hexagon can be divided into 6 congruent equilateral triangles, like the one shown with the bold edges. Each of these is made up of 9 of the small equilateral triangles of which 2 are shaded.]

Extension problem

22.1 The solution above takes it for granted that all the small triangles in the diagram are congruent. This may seem obvious from the diagram but diagrams can be misleading. So a complete solution would need to include a proof that these triangles are all congruent. Can you give a proof of this?

23. Peter wrote a list of all the numbers that could be produced by changing one digit of the number 200. How many of the numbers in Peter's list are prime? A 0 B 1 C 2 D 3 E 4

Solution: **A**

If the hundreds or tens digit of 200 is changed, but the units digit is unchanged, the resulting number is, like 200, a multiple of 10, and so cannot be prime. So we need only consider the 9 numbers that we can get by changing the units digit. Of these, we can see immediately that 202, 204, 206 and 208 are all divisible by 2 and 205 is divisible by 5. So none of them is prime. This just leaves 201, 203, 207 and 209. Now $201 = 3 \times 67$, $203 = 7 \times 29$, $207 = 3 \times 3 \times 23$ and $209 = 11 \times 19$. So none of them is prime. So none of the numbers in Peter's list are prime.

Extension problems

In this problem it was not difficult to check by direct calculations that none of the numbers in the list 201, 202…, 209 is prime. However, as numbers get larger and larger it becomes harder and harder to check whether they are prime. There are some short cuts which we explain below. But first try the next two problems.

23.1 How many of the numbers 2001, 2002, 2003, 2004, 2005, 2006, 2007, 2008, 2009 are prime?

23.2 How many of the numbers 20001, 20002, 20003, 20004, 20005, 20006, 20007, 20008 and 20009 are prime?

We note first that even numbers, other than 2, are not prime. So none of the numbers 200..000, 200..002, 200..004, 200..006, 200..008, with any number of 0s, is prime. Also numbers of the form 200..005 are always divisible by 5 and so are not prime. This just leaves numbers of the form 200..001, 200..003, 200..007 and 200..009 to be considered.

There are some short cuts we can use in these cases. You may already know that:

An integer is divisible by 3 *if and only if the sum of its digits is divisible by* 3, and *an integer is divisible by* 9 *if and only if the sum of its digits is divisible by 9*

23.3 If you have not seen this before, work out why it is true.

The sum of the digits of 200..001 is 3, so numbers of the form 200..001 are always divisible by 3. Similarly, as the sum of the digits of numbers of the form 200..007 is 9, these numbers are always divisible by 9. So none of these numbers is prime. This just leaves the numbers of the forms 200..003 and 200..009 which may or may not be prime.

23.4 Show that all the numbers of the form 200..009, with an *odd* number of 0s, are divisible by 11.

This can be shown by a direct calculation. It also follows from a standard test for divisibility by 11 based on the digits of a number.

23.5 Find out what the digits test for divisibility by 11 is (from your teacher, or from the web).

In the solution to Question 23 we noted that 203 is divisible by 7. If you tackled Problem 23.1, you may have discovered that $2009 = 7 \times 287$ and so it is also divisible by 7.

23.6 Investigate which numbers of the form 200..003 and which numbers of the form 200..009 are divisible by 7. You may find some patterns which lead to conjectures about general results. Try to prove that your conjectures are correct.

If you are successful in solving problem 23.6, you will have found some theorems about which numbers of the forms 200..003 and 200..009 are divisible by 7 and hence are not prime. Unfortunately, numbers of these forms that are not divisible by 7 may or may not be prime. So there are still lots of cases that have to be checked separately.

24. After playing 500 games, my success rate at Spider Solitaire is 49%. Assuming that I win every game from now on, how many extra games do I need to play in order that my success rate increases to 50%?

A 1 B 2 C 5 D 10 E 50

Solution: **D**

Since I have won 49% of my first 500 games, so far I have won $\frac{49}{100} \times 500 = 49 \times 5 = 245$ games. So I have lost $500 - 245 = 255$ games. I need now to win enough games so that I have won as many as I have lost. So, assuming I win every game from now on, I need to win $255 - 245 = 10$ more games.

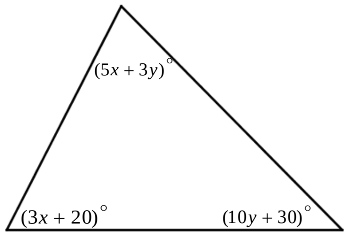
Extension problem

24.1 Assume that instead of winning every game after the first 500 games, I now lose every third game, so that after 500 games my results are Win, Win, Loss, Win, Win, Loss, … . What is the least number of extra games I need to play in order that my success rate becomes at least 50%?

25. The interior angles of a triangle are $(5x+3y)^\circ$, $(3x+20)^\circ$ and $(10y+30)^\circ$, where *x*, *y* are positive integers.

What is the value of $x+y$?

A 15 B 14 C 13 D 12 E 11



Solution: **A**

The angles of a triangle add up to 180°. So $(5x+3y)+(3x+20)+(10y+30)=180$. This gives $8x+13y+50=180$. Therefore $8x+13y=130$, and so

$$8x=130-13y=13(10-y). \qquad (*)$$

It follows that 8*x* is a multiple of 13 and hence, as 8 and 13 have no common factors, it must be that *x* is a multiple of 13. So $x=13a$, where *a* is a positive integer. It then follows from (*) that

$$8(13a)=13(10-y)$$

and so, cancelling the factor 13

$$8a=10-y,$$

and hence $10-y$ is a positive number which is a multiple of 8. The only positive value of *y* for which $10-y>0$ and $10-y$ is a multiple of 8, is $y=2$, which gives $a=1$ and hence $x=13$. So $x+y=13+2=15$.

Notes

In this problem we have just one equation, (*), with two unknowns. If there were no restrictions on the values of *x* and *y*, this equation would have infinitely many solutions. However, here we are told that *x* and *y* are both positive integers. This additional information means that there is just one solution which we found above.

The term *Diophantine equation* is used for equations where we are only interested in positive integer solutions. The name comes from Diophantus of Alexandria who is believed to have lived around 250 AD, and whose influential book *Arithmetica* discussed problems of this type.

There are many well known Diophantine equations. For example, the fact that $\sqrt{2}$ is an irrational number is equivalent to the statement that the Diophantine equation $x^2-2y^2=0$ has no solutions, as this equation may be rearranged as $\left(\frac{x}{y}\right)^2=2$.

Pythagorean triples are positive integer solutions of the Diophantine equation $x^2+y^2=z^2$. The French mathematician Pierre de Fermat (1601-1665) conjectured that none of the Diophantine equations $x^n+y^n=z^n$, where *n* is a positive integer greater than 2, has a solution where *x*, *y*, and *z* are positive integers. It was not until 1994 that Andrew Wiles proved that Fermat's "Last Theorem" was correct. Wiles' proof uses some very sophisticated mathematics – the theory of elliptic curves and modular forms.