UK INTERMEDIATE MATHEMATICAL CHALLENGE

THURSDAY 1ST FEBRUARY 2018

Organised by the **United Kingdom Mathematics Trust**
from the School of Mathematics, University of Leeds

SOLUTIONS LEAFLET

This solutions leaflet for the IMC is sent in the hope that it might provide all concerned with some alternative solutions to the ones they have obtained. It is not intended to be definitive. The organisers would be very pleased to receive alternatives created by candidates.

For reasons of space, these solutions are necessarily brief. Extended solutions, and some exercises for further investigation, can be found at:

http://www.ukmt.org.uk/

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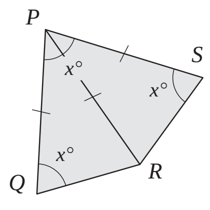
1. **B** The positive sums of cubes less than the largest option, 36, are $0^3 + 1^3 = 1$; $1^3 + 2^3 = 1 + 8 = 9$ and $2^3 + 3^3 = 8 + 27 = 35$. Hence the correct answer is 9.

2. **B** First note that 1 is not a prime. Of the other integers, 11 is prime; $111 = 3 \times 37$, so is not prime; $1111 = 11 \times 101$, so is not prime.

3. **C** The total face value of the £5 notes is £5 × 440 million = £2200 million = £2 200 000 000.

4. **D** As $PQ = PR$, $\angle PRQ = \angle PQR = x°$. Also, as $PR = PS$, $\angle PRS = \angle PSR = x°$. Therefore, $\angle QRS = (2x)°$. The sum of the interior angles of quadrilateral $PQRS$ is 360°. Therefore $x + x + 2x + x = 360$, so $5x = 360$. Hence $x = 72$.

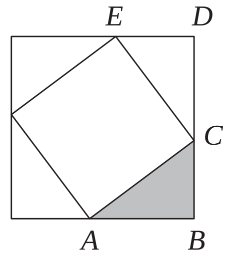


5. **B** The required fraction is $\frac{26}{206} = \frac{13}{103} \approx \frac{13}{104} = \frac{1}{8}$.

6. **A** The number of claps achieved per second is $\frac{1020}{60} = \frac{102}{6} = \frac{51}{3} = 17$.

7. **B** The two-digit squares are 16, 25, 36, 49, 64, 81. The products of their digits are 6, 10, 18, 36, 24, 8, respectively. So exactly one two-digit square has the required property, namely 49.

8. **A** First note that the length of each side of the larger square is 7 cm and that of the smaller square is 5 cm. So their areas are 49 cm^2 and 25 cm^2 respectively. Let $\angle BAC = x°$. As the interior angles of a triangle sum to 180° and $\angle ABC = 90°$, $\angle ACB = (180 - 90 - x)° = (90 - x)°$. The angles on a straight line sum to 180°, so $\angle ECD = [180 - 90 - (90 - x)]° = x°$.

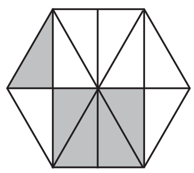


Consider triangles ABC and CDE. Note that $\angle BAC = \angle DCE = x°$; $\angle ABC = \angle CDE = 90°$; $AC = CE = 5$ cm. Therefore the two triangles are congruent (AAS). By a similar method, it may be shown that all four triangles which lie between the two squares are congruent to each other.
So the area of the shaded triangle is $\frac{1}{4}(49 - 25)$ cm^2 $= 6$ cm^2.

9. **E** We are given that $\frac{3}{10} < \frac{n}{20} < \frac{2}{5}$. Therefore $\frac{6}{20} < \frac{n}{20} < \frac{8}{20}$. So, as n is an integer, $n = 7$.

10. **E** The difference of two squares $a^2 - b^2$ factorises to $(a + b)(a - b)$. Therefore for it to be possible to express an integer as the difference of two squares, it must be possible to factorise the integer in the form $(a + b)(a - b)$, where a and b are both integers.
Note that $5 = (3 + 2)(3 - 2)$; $7 = (4 + 3)(4 - 3)$; $8 = (3 + 1)(3 - 1)$; $9 = (5 + 4)(5 - 4)$.
So 5, 7, 8, 9 can all be written as the difference of two squares.
(*It is left to the reader to show that 10 cannot be the difference of two squares as it cannot be written in the form* $(a + b)(a - b)$ *where a and b are both integers.*)

11. **B** The diagram shows that the hexagon may be divided into twelve congruent triangles, of which five are shaded. Let the area of each of these triangles be a cm^2. Then $5a = 20$, so $a = 4$. Hence the area of the hexagon, in cm^2, is $12a = 12 \times 4 = 48$.

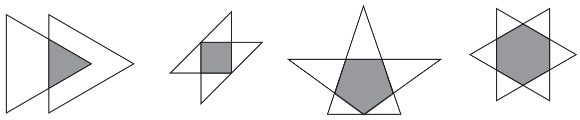


12. **E** When the given calculations are performed on Harry's calculator, they will be equal to: A 97×79; B 98×78; C 369×147; D 321×123; E 357×753.
All of these, except E, are equal to the intended calculations.

13. **C** Let the length of each edge of the rhombus be x. Then, as the rhombus and the smaller regular hexagon have an edge in common, the length of each edge of this hexagon is also x. So the length of each edge of the larger regular hexagon is $2x$. The two regular hexagons are similar. Therefore the ratio of their areas is the square of the ratio of their edge-lengths, that is $1^2 : 2^2 = 1 : 4$.

14. **C** $\frac{10}{9} + \frac{9}{10} = \frac{100 + 81}{90} = \frac{181}{90} = 2 + \frac{1}{90} = 2.0\dot{1}$.

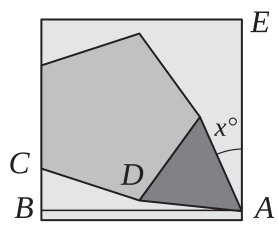
15. E The diagrams show that all four options could be the shape of the region where two triangles overlap.



16. C The sum of the interior angles of a triangle is 180°. So the three unshaded sectors inside the triangle have a total area equal to one half of the area of a circle of radius 1. Therefore the total shaded area is equal to the area of two and a half circles of radius 1, so the required area equals $\frac{5}{2} \times \pi \times 1^2 = \frac{5\pi}{2}$.

17. D The value of 'abc' equals $100a + 10b + c$. We need to find integers of the form 'abc' such that 'cba' $=$ 'abc' $+ 99$.
So $100c + 10b + a = 100a + 10b + c + 99$, that is $99c = 99a + 99$. This condition simplifies to $c = a + 1$. So there are 10 such integers of the form '1b2', 10 of the form '2b3', 10 of the form '3b4', …, and 10 of the form '8b9'. Therefore the required number is $8 \times 10 = 80$.

18. A In the diagram, line segment AB is parallel to the base of the square.
First note that each exterior angle of a regular pentagon equals $360° \div 5 = 72°$. So each interior angle of a regular pentagon equals $180° - 72° = 108°$.
Therefore obtuse angle $CDA = (60 + 108)° = 168°$ and reflex angle $CDA = (360 - 168)° = 192°$. The sum of the interior angles of a quadrilateral is 360°. Therefore, in quadrilateral $ABCD$, $\angle BAD = (360 - 192 - 72 - 90)° = 6°$. Angle BAE is a right angle. So $(6 + 60 + x)° = 90°$. Therefore $x = 90 - 66 = 24$.



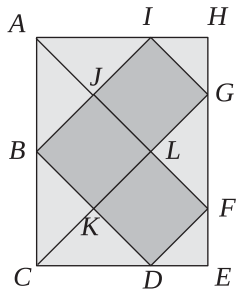
19. D As the three rectangles have equal areas: $xy = (x + 4)(y - 3) = (x + 8)(y - 4)$.
So $xy = xy - 3x + 4y - 12 = xy - 4x + 8y - 32$.
Therefore $-3x + 4y - 12 = 0$ and $-4x + 8y - 32 = 0$.
Rearranging: $3x - 4y = -12$ … (1) and $4x - 8y = -32$ … (2).
$2 \times (1) - (2)$ gives: $6x - 4x = -24 - (-32) = 8$. So $x = 4$.
Substituting for x in (1): $12 - 4y = -12$. So, $4y = 24$, that is $y = 6$.
So $x + y = 4 + 6 = 10$.

20. E First note that $72 = 8 \times 9$. So all multiples of 72 are multiples of 8 and of 9. As the only digits available are 0 and 1, the required integer is a multiple of the smallest power of 10 which is a multiple of 8, namely 1000. So the last three digits of the required integer are 000. Note also that this integer must be a multiple of 9, so the sum of its digits is a multiple of 9. Therefore, the smallest multiple of 9 whose digits are 1 and 0, and which is also a multiple of 8, is 111 111 111 000. This integer has 12 digits.

21. D Clearly, there are no values of x for which $x = x + 6$. So the possible values of x will be solutions of the equation $x = x^2$ or the equation $x + 6 = x^2$.
Consider the first equation: $x = x^2$. This simplifies to $x(1 - x) = 0$. So $x = 0$ or $x = 1$.
The equation $x + 6 = x^2$ simplifies to $x^2 - x - 6 = 0$. So $(x - 3)(x + 2) = 0$.
Therefore $x = 3$ or $x = -2$. So there are four different values of x.

22. E Let the area of each of the three squares be S. Then the areas of triangles AJI, AJB, BKC, CKD and GLF are all equal to $\frac{1}{2}S$. Also, the areas of triangles DEF and GHI are both equal to $\frac{1}{4}S$. So the total area inside the rectangle, but outside the three squares, is $5 \times \frac{1}{2}S + 2 \times \frac{1}{4}S = 3S$. Therefore the area of the rectangle is $6S = 6 \times 2 \times 2 = 24$.

(*It is left as an exercise for the reader to show that the height and width of the rectangle are* $4\sqrt{2}$ *and* $3\sqrt{2}$ *respectively and therefore that its area is* $4\sqrt{2} \times 3\sqrt{2} = 24$.)

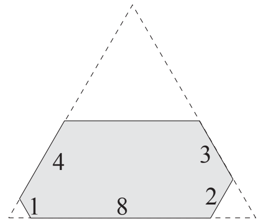


23. A The diagram shows that the hexagon is an equilateral triangle of side 11 which has an equilateral triangle of side 1 removed from the bottom left corner, an equilateral triangle of side 2 removed from the bottom right corner and an equilateral triangle of side 6 removed from the top.

Using the formula $\Delta = \frac{1}{2}ab \sin C$ for the area of a triangle, the area of an equilateral triangle of side l is

$\frac{1}{2} \times l \times l \times \sin 60° = \frac{1}{2} \times l^2 \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}l^2$. So the area of the hexagon is

$\frac{\sqrt{3}}{4}\left(11^2 - 6^2 - 1^2 - 2^2\right) = \frac{\sqrt{3}}{4}\left(121 - 36 - 1 - 4\right) = \frac{\sqrt{3}}{4} \times 80 = 20\sqrt{3}$.

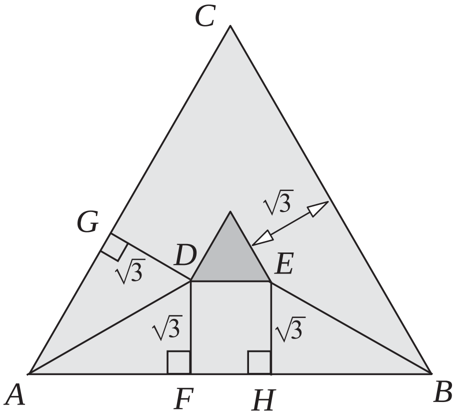


24. B As the mode of the list is 5, at least two of the integers in the list are 5. The median of the list is 5, so when the list is written in ascending order the third integer is 5. So, since the range of the list is 5, the two possible lists, in ascending order, are $x, y, 5, 5, x+5$ or $x, 5, 5, y, x+5$, where x and y are positive integers and $x \leqslant y \leqslant x+5$. The mean of the integers is 5, so their sum is $5 \times 5 = 25$, for each possible list. Therefore $x + y + 5 + 5 + x + 5 = 25$. So $2x + y = 10$. The positive integer solutions of this Diophantine equation are (1, 8), (2, 6), (3, 4), (4, 2). Note that neither x nor y equals 5, so there are exactly two 5s in the list. Of the solutions, neither (1, 8) nor (4, 2) satisfy the condition $x \leqslant y \leqslant x + 5$.

So the only two possible lists are 2, 5, 5, 6, 7 and 3, 4, 5, 5, 8.

25. D In the diagram shown, F is the foot of the perpendicular from D to AB, G is the foot of the perpendicular from D to AC and H is the foot of the perpendicular from E to AB. So $DF = DG = EH = \sqrt{3}$. Consider triangles AFD and AGD. Edge AD is common to both triangles, $DF = DG$ and angles AFD and AGD are both right angles. So the triangles are congruent (RHS). Therefore $\angle DAF = \angle DAG = 60° \div 2 = 30°$.

So in triangle AFD, $\tan 30° = \frac{DF}{AF}$.



Therefore $AF = \sqrt{3} \div (1/\sqrt{3}) = \sqrt{3} \times \sqrt{3} = 3$. The same argument shows that $BH = 3$. It is clear that $FH = DE$, so $AB - DE = AF + BF = 3 + 3 = 6$. Therefore the difference between the lengths of the edges of the two triangles equals 6.