2016

Intermediate Mathematical Challenge

Organised by the United Kingdom Mathematics Trust

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Solutions and investigations

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some exercises for further investigation. We welcome comments on these solutions. Please send them to enquiry@ukmt.org.uk.

The Intermediate Mathematical Challenge (IMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the IMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. So for each question we have included a complete solution which does not use the fact that one of the given alternatives is correct. Thus we have aimed to give full solutions with all steps explained. We therefore hope that these solutions can be used as a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the Intermediate Mathematical Olympiad and similar competitions).

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
B	D	B	E	A	C	C	E	C	E	D	D	B	D	A	B	B	C	B	D	A	E	B	C	D

1. What is the value of $6102 - 2016$?

A 3994 B 4086 C 4096 D 4114 E 4994

SOLUTION **B**

The direct method is to do the subtraction:

$$\begin{array}{rcccc} & 6 & 1 & 0 & 2 \\ - & 2 & 0 & 1 & 6 \\ \hline & 4 & 0 & 8 & 6 \\ \hline \end{array}$$

NOTE

Note that in the context of the IMC where we just need to decide which of the given options is correct, it isn't really necessary to do this subtraction. The digits of both 6102 and 2016 add up to 9. So both 6102 and 2016 are multiples of 9. It follows that their difference is also a multiple of 9. So the digits of the difference add up to a multiple of 9. Of the given options, 4086 is the only one which satisfies this condition ($4 + 0 + 8 + 6 = 18$). So assuming, as you are entitled to, that one of the options is correct, you can deduce that B is the correct option.

Alternatively, if you do the subtraction, the fact that the sum of the digits of 4086 is a multiple of 9, serves as a useful check.

FOR INVESTIGATION

1.1 Prove that the following facts, used in the above Note, are correct.

(a) The test for whether an integer is a multiple of 9 is whether the sum of its digits is a multiple of 9.

(b) If m and n are both integers that are multiples of 9, then $m - n$ is also a multiple of 9.

1.2 The test for whether an integer is divisible by 9, as given above, can be extended: Given a positive integer n, both n and the sum of the digits of n have the same remainder when they are divided by 9. Show that this is correct.

1.3 Find the remainders when the following numbers are divided by 9.

(a) 6 666 666,

(b) 7 777 777,

(c) 8 888 888.

1.4 You are told that exactly one of the following calculations is correct. Which is it?

(a) $987\,654\,321 - 123\,456\,789 = 864\,196\,532$,

(b) $987\,654\,321 - 123\,456\,789 = 863\,197\,532$,

(c) $987\,654\,321 - 123\,456\,789 = 864\,197\,532$.

2. Which of the following fractions is closest to 1?

A $\frac{7}{8}$ B $\frac{8}{7}$ C $\frac{9}{10}$ D $\frac{10}{11}$ E $\frac{11}{10}$

SOLUTION **D**

We tackle this question by calculating the differences between the given options and 1.

We have

$$1 - \frac{7}{8} = \frac{1}{8}, \quad \frac{8}{7} - 1 = \frac{1}{7}, \quad 1 - \frac{9}{10} = \frac{1}{10}, \quad 1 - \frac{10}{11} = \frac{1}{11}, \text{ and } \frac{11}{10} - 1 = \frac{1}{10}.$$

The smallest of these differences is $\frac{1}{11}$. Therefore, of the given options, $\frac{10}{11}$ is closest to 1.

3. How many of these five expressions give answers which are *not* prime numbers?

$1^2 + 2^2$	$2^2 + 3^2$	$3^2 + 4^2$	$4^2 + 5^2$	$5^2 + 6^2$

A 0 B 1 C 2 D 3 E 4

SOLUTION **B**

We have

$$\begin{aligned} 1^2 + 2^2 &= 1 + 4 = 5, \\ 2^2 + 3^2 &= 4 + 9 = 13, \\ 3^2 + 4^2 &= 9 + 16 = 25, \\ 4^2 + 5^2 &= 16 + 25 = 41 \\ \text{and} \quad 5^2 + 6^2 &= 25 + 36 = 61. \end{aligned}$$

Of these answers 5, 13, 41 and 61 are prime numbers, but 25 is not a prime number.

Therefore just one of the given options does not work out to be a prime number.

FOR INVESTIGATION

3.1 Show that there do not exist three consecutive positive integers n for which $n^2 + (n+1)^2$ is a prime number.

3.2 Find the smallest positive integer n for which none of the numbers $n^2 + (n+1)^2$, $(n+1)^2 + (n+2)^2$ and $(n+2)^2 + (n+3)^2$ is a prime number.

4. Amrita is baking a cake today. She bakes a cake every fifth day.

How many days will it be before she next bakes a cake on a Thursday?

A 5 B 7 C 14 D 25 E 35

SOLUTION **E**

METHOD 1

Amrita bakes a cake again after 5, 10, 15, 20, 25, 30, 35 days and so on. Note that the IMC took place on a Thursday and so succeeding Thursdays come after 7, 14, 21, 28, 35 days and so on. The first number in both of these lists is 35. So Amrita next bakes a cake on a Thursday after 35 days.

METHOD 2

Amrita bakes a cake after n days just when n is a multiple of 5. As the IMC took place on a Thursday, it is another Thursday in n days time just when n is a multiple of 7. So Amrita bakes a cake on a Thursday in n days time just when n is a multiple of both 5 and of 7. The lowest such positive integer n is the lowest common multiple of 5 and 7. The lowest common multiple of 5 and 7 is $5 \times 7 = 35$. So Amrita next bakes a cake on a Thursday after 35 days.

FOR INVESTIGATION

4.1 Amrita makes jam and chutney today. She makes jam every 12 days and makes chutney every 20 days. After how many days will she again make both jam and chutney?

5. When travelling from London to Edinburgh by train, you pass a sign saying "Edinburgh 200 miles". Then, $3\frac{1}{2}$ miles later, you pass another sign saying "Half way between London and Edinburgh".

How many miles is it by train from London to Edinburgh?

A 393 B $396\frac{1}{2}$ C 400 D $403\frac{1}{2}$ E 407

SOLUTION **A**

After travelling a further $3\frac{1}{2}$ miles towards Edinburgh after passing the "Edinburgh 200 miles" sign, the train is $(200 - 3\frac{1}{2})$ miles $= 196\frac{1}{2}$ miles from Edinburgh. As the train is now half way between London and Edinburgh, the distance from London to Edinburgh is $(2 \times 196\frac{1}{2})$ miles $= 393$ miles .

6. One third of the animals in Jacob's flock are goats, the rest are sheep. There are twelve more sheep than goats.

How many animals are there altogether in Jacob's flock?

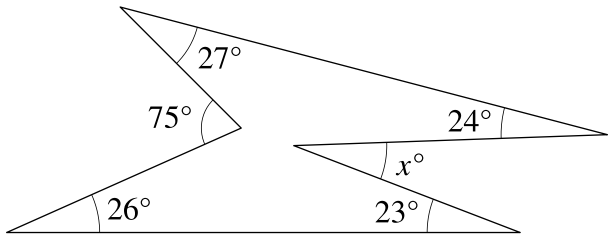
A 12 B 24 C 36 D 48 E 60

SOLUTION **C**

One third of the animals in Jacob's flock are goats. Therefore two thirds of these animals are sheep. Therefore, the 12 more sheep than goats amount to one third of the flock. Since 12 animals make up one third of the flock, there are $3 \times 12 = 36$ animals in Jacob's flock.

7. In the diagram, what is the value of x?

A 23 B 24 C 25
D 26 E 27



SOLUTION **C**

The sum of the interior angles of a polygon with n sides is $(n - 2) \times 180°$. So the sum of the interior angles of the hexagon in the diagram is $(6 - 2) \times 180° = 4 \times 180° = 720°$.

The sum of the angles at a point is 360°. Therefore interior angle of the hexagon adjacent to the exterior angle of 75° is $(360 - 75)°$, and the interior angle adjacent to the exterior angle of $x°$ is $(360 - x)°$. Since the sum of the interior angles of the hexagon is 720°, we have

$$24 + 27 + (360 - 75) + 26 + 23 + (360 - x) = 720.$$

We can rearrange the left hand side of this equation to obtain

$$(24 + 27 + 26 + 23) - 75 + (360 + 360) - x = 720.$$

It follows that $100 - 75 + 720 - x = 720$. Hence $25 - x = 0$. Therefore $x = 25$.

FOR INVESTIGATION

7.1 An alternative method is to consider the angles turned through anticlockwise to get from each edge of the hexagon to the next as you go round it anticlockwise, starting with the angle $x°$.

Check that the total angle, measured anticlockwise, turned through is $(x - 24 - 27 + 75 - 26 - 23)°$. Use this to deduce that $x = 25$.

7.2 Prove that the sum of the interior angles of a polygon with n sides is $(n - 2) \times 180°$.

7.3 How many degrees are there in each interior angle of a regular polygon with n sides?

7.4 What is the least value of n such that the number of degrees in each interior angle of a regular polygon with n sides is *not* an integer?

8. What is the value of $2.017 \times 2016 - 10.16 \times 201.7$?

A 2.016 B 2.017 C 20.16 D 2016 E 2017

Solution **E**

To avoid a lot of complicated arithmetic, we exploit the facts that $2.017 = \frac{201.7}{100}$ and $10.16 = \frac{1016}{100}$. Then we take out the common factor $\frac{201.7}{100}$. This gives

$$\begin{aligned} 2.017 \times 2016 - 10.16 \times 201.7 &= \frac{201.7}{100} \times 2016 - \frac{1016}{100} \times 201.7 \\ &= \frac{201.7}{100} \times (2016 - 1016) \\ &= \frac{201.7}{100} \times 1000 \\ &= 201.7 \times 10 \\ &= 2017. \end{aligned}$$

9. The world's fastest tortoise is acknowledged to be a leopard tortoise from County Durham called Bertie. In July 2014, Bertie sprinted along a 5.5 m long track in an astonishing 19.6 seconds.

What was Bertie's approximate average speed in km per hour?

A 0.1 B 0.5 C 1 D 5 E 10

Solution **C**

It took Bertie 19.6 seconds to travel 5.5 m. So Bertie went $\frac{5.5}{19.6}$ m in 1 second. There are 3600 seconds in one hour. Therefore Bertie travelled

$$\left(\frac{5.5}{19.6} \times 3600\right) \text{m}$$

in one hour.

There are 1000 m in 1 km. Therefore Bertie's average speed in km per hour was

$$\frac{5.5}{19.6} \times \frac{3600}{1000} = \frac{5.5}{19.6} \times \frac{18}{5}.$$

This is approximately equal to

$$\frac{5}{18} \times \frac{18}{5} = 1.$$

Therefore Bertie's approximate average speed was 1 km per hour.

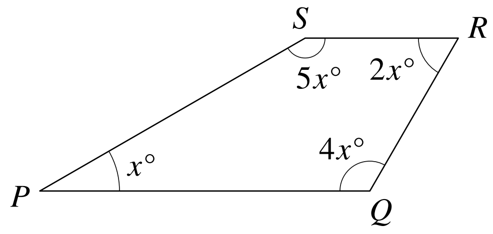
10. The angles of a quadrilateral taken in order are $x°$, $5x°$, $2x°$ and $4x°$.

Which of the following is the quadrilateral?

A kite B parallelogram C rhombus D arrowhead E trapezium

Solution **E**

We let the vertices of the quadrilateral be P, Q, R, S with the angles as shown in the diagram.



The sum of the interior angles of a quadrilateral is 360°. Therefore, $x + 5x + 2x + 4x = 360$. That is, $(x + 5x) + (2x + 4x) = 360$. Since $x + 5x = 2x + 4x$, it follows that $x + 5x = 2x + 4x = 180$.

Hence $\angle SPQ + \angle RSP = 180°$. Therefore the sides PQ and SR of the quadrilateral are parallel. Since $x + 4x = 5x \neq 180$, we have $\angle SPQ + \angle PQR \neq 180°$. Therefore the sides PS and QR are not parallel.

We therefore see that the quadrilateral has one pair of opposite sides that are parallel and one pair of opposite sides that are not parallel. Therefore the quadrilateral is a trapezium.

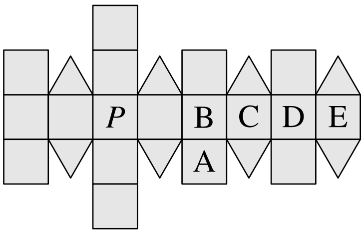
For investigation

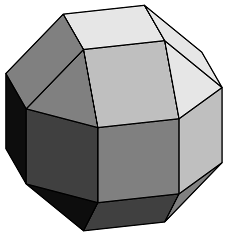
10.1 Note that in the above solution we showed that the quadrilateral is a trapezium without finding the value of x. However, the information given does enable us to find the value of x. Find x and hence the sizes of the angles in the trapezium.

10.2 You are given that the angles of a quadrilateral taken in order are $4x°$, $5x°$, $4x°$ and $5x°$. Can you say which of the shapes specified as the options in Question 10 this quadrilateral is?

10.3 You are given that the angles of a quadrilateral taken in order are $2x°$, $3x°$, $7x°$ and $6x°$. Can you say which type of quadrilateral it is?

11. The net shown consists of squares and equilateral triangles. The net is folded to form a rhombicuboctahedron, as shown.



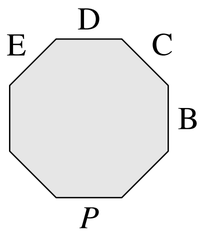


When the face marked P is placed face down on a table, which face will be facing up?

A B C D E

Solution **D**

The diagram shows a vertical cross-section through the centre of the rhombicuboctahedron when it is placed so that the face marked P is face down.



The cross-section cuts through a ring of eight square faces. These are the eight square faces that form the central band of the net. They are labelled as shown. We see from the diagram that the face D will be facing up.

12. The sum of two numbers a and b is 7 and the difference between them is 2.

What is the value of $a \times b$?

A $8\frac{1}{4}$ B $9\frac{1}{4}$ C $10\frac{1}{4}$ D $11\frac{1}{4}$ E $12\frac{1}{4}$

Solution **D**

We have $a + b = 7$. Also, depending on whether $a > b$ or $b > a$, the value of $a - b$ is either $+2$ or -2. In either case, $(a - b)^2 = 4$, and so $(a + b)^2 - (a - b)^2 = 49 - 4 = 45$. Now

$$\begin{aligned}(a+b)^2 - (a-b)^2 &= (a^2 + 2ab + b^2) - (a^2 - 2ab + b^2) \\ &= 4ab.\end{aligned}$$

Hence $4ab = 45$. It follows that $a \times b = \frac{45}{4} = 11\frac{1}{4}$.

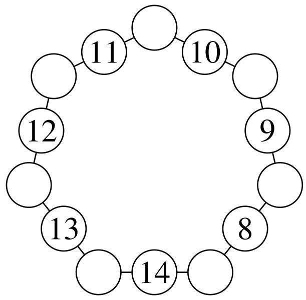
Note

In the above solution we were able to find the value of ab without first finding the values of a and b. An alternative method would be to first find a and b and then ab. Exercise 12.1 asks you to do this.

For investigation

12.1 Assume that $a > b$. So $a + b = 7$ and $a - b = 2$. Find the values of a and b. Hence verify that $a \times b = 11\frac{1}{4}$. Check that this is also true in the case that $b > a$.

13. The diagram shows a heptagon with a line of three circles on each side. Each circle is to contain exactly one number. The numbers 8 to 14 are distributed as shown and the numbers 1 to 7 are to be distributed to the remaining circles. The total of the numbers in each of the lines of three circles is to be the same.



What is this total?

A 18 B 19 C 20 D 21 E 22

SOLUTION **B**

Each of the integers from 1 to 7 occurs in exactly two of the seven lines of three circles. Each of the integers from 8 to 14 occurs in exactly one of these lines.

Therefore, if we add up all the totals of the numbers in the seven lines, we are adding each of the integers from 1 to 7 twice, and each of the integers from 8 to 14 once. It follows that the sum of the seven line totals is

$$\begin{aligned} 2 \times (1+2+3+4+5+6+7) + (8+9+10+11+12+13+14) &= 2 \times 28 + 77 \\ &= 56 + 77 \\ &= 133. \end{aligned}$$

Since the total for each of the seven lines is the same, the total of the numbers in one line is $133 \div 7 = 19$.

In the context of the IMC we can stop here. We have shown that, if the numbers can be placed so that the total of the numbers in each of the lines of three circles is the same, then this common total is 19. So, assuming, as we are entitled to, that one of the options is correct, we deduce that B is the correct option.

For a complete solution, however, it is necessary to show that the numbers can be placed so that the total of the numbers in each line is 19. You are asked to do this in Exercise 13.1.

FOR INVESTIGATION

13.1 (a) Show that it is possible to arrange the integers from 1 to 7 in the remaining circles so that the total of the numbers in each line is 19.

(b) In how many different ways can this be done?

14. Tegwen has the same number of brothers as she has sisters. Each one of her brothers has 50% more sisters than brothers.

How many children are in Tegwen's family?

A 5 B 7 C 9 D 11 E 13

Solution **D**

Method 1

Tegwen has equal numbers of brothers and sisters. Consider one of her brothers, say Dafydd. Dafydd has one more sister than Tegwen has, namely Tegwen. He has one fewer brother than Tegwen has, namely himself.

It follows that Dafydd has 2 more sisters than he has brothers. He has 50% more sisters than brothers. Since 2 sisters is 50% of the number of his brothers, Dafydd has 4 brothers. Hence, he has 6 sisters. The children in the family are Dafydd's sisters and brothers and Dafydd. So there are $6 + 4 + 1 = 11$ children in the family.

Method 2

Suppose that Tegwen has n brothers. Since she has equal numbers of brothers and sisters, she also has n sisters.

Consider one of Tegwen's brothers, say Dafydd. All of Tegwen's sisters are also Dafydd's sisters, and Dafydd also has Tegwen as one of his sisters. So Dafydd has $n + 1$ sisters. All of Tegwen's brothers, other than Dafydd himself, are also Dafydd's brothers. So Dafydd has $n - 1$ brothers.

Since Dafydd has 50% more sisters than brothers, it follows that

$$\frac{n+1}{n-1} = \frac{150}{100} = \frac{3}{2}$$

and therefore

$$2(n + 1) = 3(n - 1),$$

that is,

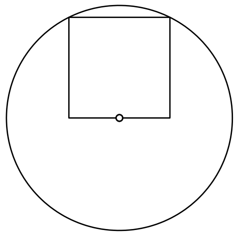
$$2n + 2 = 3n - 3,$$

from which it follows that

$$n = 5.$$

Therefore Tegwen has 5 brothers and 5 sisters. So, including Tegwen, there are $5 + 5 + 1 = 11$ children in her family.

15. The circle has radius 1 cm. Two vertices of the square lie on the circle. One edge of the square goes through the centre of the circle, as shown.

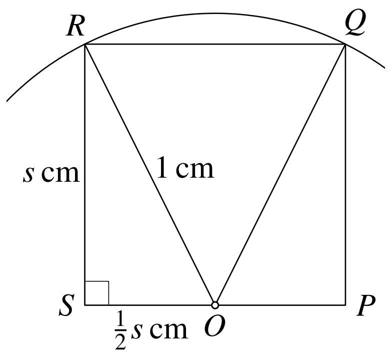


What is the area of the square?

A $\frac{4}{5}\,\text{cm}^2$ B $\frac{\pi}{5}\,\text{cm}^2$ C $1\,\text{cm}^2$ D $\frac{\pi}{4}\,\text{cm}^2$ E $\frac{5}{4}\,\text{cm}^2$

Solution **A**

Let the vertices of the square be P, Q, R and S, as shown in the diagram. Let O be the centre of the circle. We also suppose that the length of the sides of the square is s cm. It follows that the area of the square is $s^2\,\text{cm}^2$.



We consider the triangles POQ and SOR. In these triangles $\angle OPQ = \angle OSR = 90°$, as they are both angles in the square; $OQ = OR$, as they are radii of the same circle; and $PQ = SR$, as they are sides of the same square.

It follows that the triangles POQ and SOR are congruent, by the RHS congruence condition. [RHS stands for Right-angle, Hypotenuse, Side.] It follows that $OP = OS = \frac{1}{2}s$ cm. Therefore, by applying Pythagoras' Theorem to the triangle SOR, we have

$$s^2 + (\tfrac{1}{2}s)^2 = 1^2,$$

that is,

$$s^2 + \tfrac{1}{4}s^2 = 1,$$

and hence

$$\tfrac{5}{4}s^2 = 1,$$

from which it follows that

$$s^2 = \tfrac{4}{5}.$$

Therefore the area of the square is $\frac{4}{5}\,\text{cm}^2$.

16. How many of the following positive integers are divisible by 24?

$2^2 \times 3^2 \times 5^2 \times 7^3$	$2^2 \times 3^2 \times 5^3 \times 7^2$	$2^2 \times 3^3 \times 5^2 \times 7^2$	$2^3 \times 3^2 \times 5^2 \times 7^2$

A 0   B 1   C 2   D 3   E 4

SOLUTION **B**

An integer n is divisible by 24 if, and only if, $n = 24 \times m$, for some integer m. Since the prime factorization of 24 is $2^3 \times 3^1$, it follows that n is divisible by 24 if, and only if $n = 2^3 \times 3^1 \times m$, for some integer m.

Another way to put this is that an integer n is divisible by 24 if, and only if, in the prime factorization of n, the exponent of 2 is at least 3, and the exponent of 3 is at least 1.

It follows from this that, of the given numbers, there is just one which is divisible by 24, namely $2^3 \times 3^2 \times 5^2 \times 7^2$.

FOR INVESTIGATION

16.1 Which of the numbers given in the above question are divisible by 375?

16.2 Which of the numbers

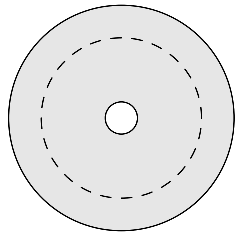
$$2^4 \times 3^4 \times 5^4, \qquad 2^5 \times 3^5 \times 5^5, \qquad 2^6 \times 3^6 \times 5^6$$

are divisible by 160?

17. The shaded region in the diagram, bounded by two concentric circles, is called an *annulus*. The circles have radii 2 cm and 14 cm.

The dashed circle divides the area of this annulus into two equal areas. What is its radius?

A 9 cm B 10 cm C 11 cm D 12 cm E 13 cm



Solution **B**

In answering this question we use the fact that the area of a circle with radius r is πr^2.

Let the radius of the dashed circle be x cm.

The area of the shaded annulus is the area inside the outer circle of radius 14 cm less the area inside the inner circle of radius 2 cm. Therefore this area is, in cm^2,

$$\pi(14^2) - \pi(2^2) = 196\pi - 4\pi = 192\pi.$$

Similarly, the area between the dashed circle and the inner circle is, in cm^2,

$$\pi x^2 - \pi(2^2) = (x^2 - 4)\pi.$$

Since the dashed circle divides the annulus into two equal areas, the area between the dashed circle and the inner circle is equal to half the area of the annulus. Half the area of the annulus is, in cm^2, $192\pi \div 2 = 96\pi$. Therefore

$$(x^2 - 4)\pi = 96\pi.$$

It follows that

$$x^2 - 4 = 96,$$

and hence that

$$x^2 = 100.$$

The number x is positive as it is the radius, in centimetres, of the dashed circle. It follows that $x = 10$. Hence the radius of the dashed circle is 10 cm.

For investigation

17.1 Suppose that the outer circle has radius r cm and the inner circle has radius s cm. Find the radius, in terms of r and s of the dashed circle that divides the area of the annulus into two equal areas.

17.2 Check that the formula that you found as your answer to 17.1 gives the correct answer to Question 17 when you put $r = 14$ and $s = 2$.

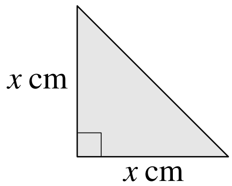
18. The sum of the areas of the squares on the sides of a right-angled isosceles triangle is $72\,\text{cm}^2$.

What is the area of the triangle?

A $6\,\text{cm}^2$ B $8\,\text{cm}^2$ C $9\,\text{cm}^2$ D $12\,\text{cm}^2$ E $18\,\text{cm}^2$

SOLUTION **C**

Let the equal sides of the right-angled isosceles triangle have length x cm. Then the squares on these sides have area $x^2\,\text{cm}^2$. Therefore, by Pythagoras' Theorem, the area of the square on the hypotenuse of the triangle is $(x^2+x^2)\,\text{cm}^2$, that is $2x^2\,\text{cm}^2$. Hence, the sum of the areas of the squares on all three sides of the triangle is $(x^2+x^2+2x^2)\,\text{cm}^2 = 4x^2\,\text{cm}^2$. It follows that $4x^2 = 72$ and hence that $x^2 = 18$.



The triangle is half of a square measuring $x\,\text{cm} \times x\,\text{cm}$. Therefore the triangle has area $\frac{1}{2}x^2\,\text{cm}^2$. It follows that the area of the triangle is $9\,\text{cm}^2$.

FOR INVESTIGATION

18.1 The sum of the areas of the squares on the sides of a right-angled triangle is $40\,\text{cm}^2$. What is the area of the square on the hypotenuse of the triangle?

18.2 The sum of the areas of the squares on the sides of a right-angled triangle is $130\,\text{cm}^2$. The difference between the lengths of the two sides adjacent to the right angle is 3 cm. What is the area of the triangle?

19. A list of positive integers has a median of 8, a mode of 9 and a mean of 10. What is the smallest possible number of integers in the list? A 5 B 6 C 7 D 8 E 9

Solution **B**

Because the list has a mode of 9, the integer 9 must occur at least twice in the list. Therefore, as the mean is 10, the list must include at least one integer which is greater than 10.

So the list includes at least three integers that are greater than 8. Because the median is 8, the list must also include at least three integers that are less than 8. It follows that there are at least six integers in the list.

We now try and find a list of six integers with the required properties. We can suppose from what we already know that the list, in increasing order, is

$$p,\ q,\ r,\ 9,\ 9,\ u$$

where $r < 8$ and $u > 10$.

For this list to have median 8, we need to have $\frac{1}{2}(r + 9) = 8$, and therefore $r = 7$. So the list, in increasing order is

$$p,\ q,\ 7,\ 9,\ 9,\ u$$

with $u > 10$.

For this list to have mode 9, no other integer can occur more than once in the list. Therefore $p \neq q$ and $q \neq 7$. So $p < q < 7$.

For this list to have mean 10, we need to have $p + q + 7 + 9 + 9 + u = 6 \times 10 = 60$, that is, $p + q + u + 25 = 60$.

Therefore we require that $p + q + u = 35$, with $p < q < 7$, and $u > 10$.

There is more than one way to choose p, q and u so as to meet these conditions. For example, the list

$$5,\ 6,\ 7,\ 9,\ 9,\ 24$$

has a median of 8, a mode of 9 and a mean of 10.

We have found a list of 6 positive integers with the given properties. We have shown that no list of fewer than 6 positive integers has these properties. We conclude that the smallest possible number of integers in the list is 6.

For investigation

19.1 How many different lists

$$p,\ q,\ r,\ s,\ t,\ u$$

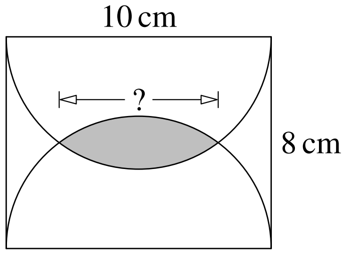
are there of six positive integers in increasing order which have a median of 8, a mode of 9 and a mean of 10?

19.2 A list of positive integers has a median of 9, a mode of 8 and a mean of 10. What is the smallest possible number of integers in the list?

20. Two semicircles are drawn in a rectangle as shown.

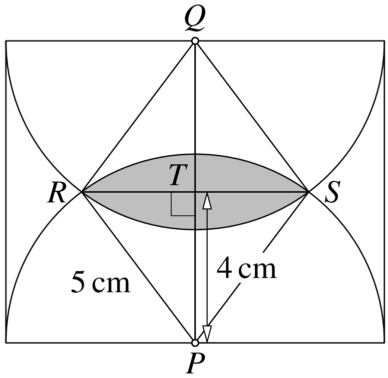
What is the width of the overlap of the two semicircles?

A 3 cm B 4 cm C 5 cm D 6 cm E 7 cm



Solution **D**

Let P, Q be the midpoints of the longer sides of the rectangle. Let R, S be the points where the semicircles meet. Let T be the point where PQ meets RS.



Each of the semicircles has radius 5 cm. Therefore PR, PS, QR and QS all have length 5 cm. Therefore $PSQR$ is a rhombus. Hence the diagonals PQ and RS bisect each other at right angles. It follows that PT and QT each have length 4 cm. Let the common length of RT and ST be x cm.

We now apply Pythagoras' Theorem to the right-angled triangle PTR. This gives $4^2 + x^2 = 5^2$, and hence $x^2 = 5^2 - 4^2 = 25 - 16 = 9$. Therefore $x = 3$.

It follows that both RT and ST have length 3 cm. Hence the length of RS is 6 cm. Therefore the width of the overlap of the two semicircles is 6 cm.

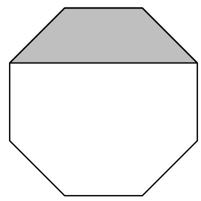
For investigation

20.1 The above solution tacitly assumes that PQ is parallel to the shorter sides of the rectangle, and so has length 8 cm. Prove that this is correct.

20.2 The above solution uses the fact that the diagonals of a rhombus bisect each other at right angles. Prove that this is correct.

21. The diagram shows a regular octagon.

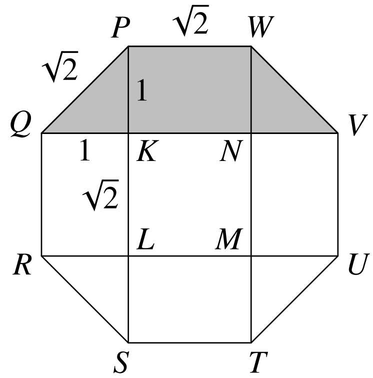
What is the ratio of the area of the shaded trapezium to the area of the whole octagon?



A 1 : 4 B 5 : 16 C 1 : 3 D $\sqrt{2}$: 2 E 3 : 8

Solution **A**

We let P, Q, R, S, T, U, V and W be the vertices of the regular octagon. Let K, L, M and N be the points where the diagonals PS and WT meet the diagonals QV and RU, as shown in the diagram.



Because $PQRSTUVW$ is a regular octagon, QKP, WNV, SLR and UMT are congruent right-angled isosceles triangles. We choose units so that the shorter sides of these triangles have length 1.

Then, by Pythagoras' Theorem, the hypotenuse of each of these triangles has length $\sqrt{1^2 + 1^2} = \sqrt{2}$. Since the octagon is regular, it follows that each of its sides has length $\sqrt{2}$.

Each of the triangles QKP, WNV, SLR and UMT, forms half of a square with side length 1, and so has area $\frac{1}{2}$.

The shaded trapezium is made up of the two triangles QKP and WNV, each of area $\frac{1}{2}$, together with the rectangle $KNWP$ which has area $\sqrt{2} \times 1 = \sqrt{2}$. Therefore the area of the shaded trapezium is $2 \times \frac{1}{2} + \sqrt{2} = 1 + \sqrt{2}$.

The octagon is made up of the four triangles QKP, WNV, SLR and UMT, each with area $\frac{1}{2}$, the four rectangles $KNWP$, $RLKQ$, $MUVN$ and $STML$, each with area $\sqrt{2}$, and the central square $KLMN$ which has area $\sqrt{2} \times \sqrt{2} = 2$. Therefore the area of the octagon is $4 \times \frac{1}{2} + 4 \times \sqrt{2} + 2 = 4 + 4\sqrt{2} = 4(1 + \sqrt{2})$.

It follows that the ratio of the area of the shaded trapezium to the area of the octagon is $1 + \sqrt{2} : 4(1 + \sqrt{2}) = 1 : 4$.

For investigation

21.1 Show how the octagon may be dissected into four quadrilaterals of equal area, one of which is the shaded trapezium. [Note: these quadrilaterals will not all be congruent.]

21.2 Show how the octagon may be dissected into four congruent hexagons.

21.3 The above solution begins with the claim that, because the octagon is regular, the triangles QKP, WNV, SLR and UMT are congruent isosceles right-angled triangles. Prove that this claim is correct.

22. In a particular group of people, some always tell the truth, the rest always lie. There are 2016 in the group. One day, the group is sitting in a circle. Each person in the group says, "Both the person on my left and the person on my right are liars."

What is the difference between the largest and smallest number of people who could be telling the truth?

A 0 B 72 C 126 D 288 E 336

SOLUTION **E**

Each truth teller must be sitting between two liars. Each liar must be sitting next to at least one truth teller and may be sitting between two truth tellers.

The largest number of truth tellers occurs when each liar is sitting between two truth tellers and each truth teller is sitting between two liars. In this case the truth tellers and liars alternate around the table. So half (that is, 1008) of the people are truth tellers and half (1008) are liars. This arrangement is possible because 2016 is even.

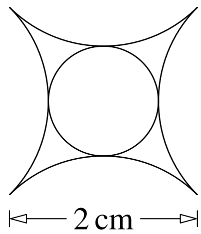
The smallest number of truth tellers occurs when each liar is sitting next to just one truth teller and so is sitting between a truth teller and a liar. In this case as you go round the table there is one truth teller, then two liars, then one truth teller, then two liars and so on. So one-third (672) of the people are truth tellers and two-thirds (1374) are liars. This arrangement is possible because 2016 is divisible by 3.

Therefore, the difference between the largest and smallest numbers of people who could be telling the truth is $1008 - 672 = 336$.

FOR INVESTIGATION

22.1 The solution above shows that, for $n = 1008$ and $n = 672$, it is possible for there to be n truth tellers and $2016 - n$ liars. Determine for which other values of n it is possible for there to be n truth tellers and $2016 - n$ liars.

23. A Saxon silver penny, from the reign of Ethelbert II in the eighth century, was sold in 2014 for £78 000. A design on the coin depicts a circle surrounded by four equal arcs, each a quarter of a circle, as shown. The width of the design is 2 cm.

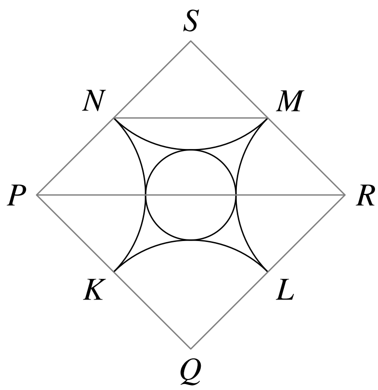


What is the radius of the small circle, in centimetres?

A $\frac{1}{2}$ B $2 - \sqrt{2}$ C $\frac{1}{2}\sqrt{2}$ D $5 - 3\sqrt{2}$ E $2\sqrt{2} - 2$

SOLUTION **B**

We see from the diagram that the quarter circles touch. [This can be proved See Exercise 23.3.] Let the points where they touch be K, L, M and N. Let the centres of the quarter circles be P, Q, R and S, arranged as shown in the diagram.



Because the circles touch, PKQ, QLR, RMS and SNP are straight lines.

Since S is the centre of the quarter circle that goes through M and N, we have $SM = SN$ and $\angle MSN = 90°$. Therefore MSN is a right-angled isosceles triangle. The length of the hypotenuse NM of this triangle is the width of the design, that is, 2 cm.

Therefore, by Pythagoras' Theorem applied to the triangle SMN, we see that SM and SN each have length $\sqrt{2}$ cm.

Similarly MR and NP both have length $\sqrt{2}$ cm. Therefore SPR is a right-angled isosceles triangle in which both SR and SP have length $2\sqrt{2}$ cm. Therefore, by Pythagoras' Theorem applied to the triangle SPR, the hypotenuse PR of this triangle has length 4 cm.

[Alternatively, we could argue that SNM and SPR are similar triangles in which SP is twice the length of SN. Therefore PR is twice the length of NM.]

The line segment PR is made up of two radii of the quarter circles with centres P and R, which have total length $2\sqrt{2}$ cm, and a diameter of the small circle. It follows that the diameter of the small circle has length $4\text{ cm} - 2\sqrt{2}$ cm. The radius of the small circle is half of this, that is, $(2 - \sqrt{2})$ cm.

FOR INVESTIGATION

23.1 Prove that the following statements, used in the above solution, are correct.

(a) RMS is a straight line.

(b) SM and SN have length $\sqrt{2}$ cm.

(c) PR has length 4 cm.

23.2 Find the area of the shape enclosed by the four quarter circles.

23.3 Deduce that the quarter circles touch from the fact that the four arcs are all quarter circles.

24. Every day, Aimee goes up an escalator on her journey to work. If she stands still, it takes her 60 seconds to travel from the bottom to the top. One day the escalator was broken so she had to walk up it. This took her 90 seconds.

How many seconds would it take her to travel up the escalator if she walked up at the same speed as before while it was working?

A 30 B 32 C 36 D 45 E 75

Solution **C**

We suppose that the distance from the bottom to the top of the escalator is d metres.

Since it takes Aimee 60 seconds to travel this distance when she stands still, the escalator moves upwards at a speed of $\frac{d}{60}$ metres per second.

Since it takes Aimee 90 seconds to walk from the bottom to the top of the escalator when it is not moving, Aimee walks up the escalator at a speed of $\frac{d}{90}$ metres per second.

It follows that when Aimee is walking at the same speed up the moving escalator she is moving at a speed of $\left(\frac{d}{60} + \frac{d}{90}\right)$ metres per second.

Now

$$\left(\frac{d}{60} + \frac{d}{90}\right) = d\left(\frac{1}{60} + \frac{1}{90}\right) = d\left(\frac{3+2}{180}\right) = d\left(\frac{5}{180}\right) = \frac{d}{36}.$$

So Aimee moves up the escalator at a speed of $\frac{d}{36}$ metres per second. At this speed it takes Aimee 36 seconds to travel the d metres up the escalator.

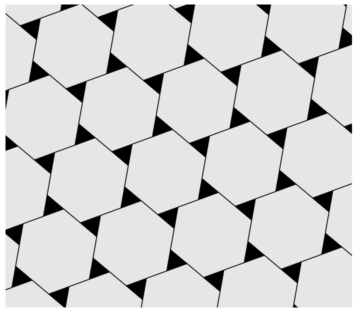
For investigation

24.1 When the escalator is not moving Aimee can run down it in 45 seconds. How long does it take Aimee to reach the bottom of the escalator from the top when she is running down at the same speed as when the escalator is not moving, and the escalator is moving upwards at its normal speed?

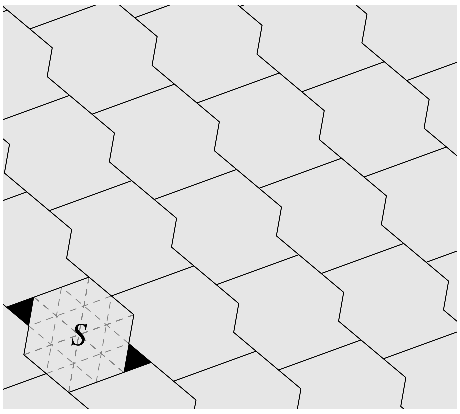
25. The tiling pattern shown uses two types of tile, regular hexagons and equilateral triangles, with the length of each side of the equilateral triangles equal to half the length of the sides of each side of the hexagons. A large number of tiles is used to cover a floor.

Which of the following is closest to the fraction of the floor that is shaded black?

A $\frac{1}{8}$ B $\frac{1}{10}$ C $\frac{1}{12}$ D $\frac{1}{13}$ E $\frac{1}{16}$



Solution **D**



We see that the entire plane can be tessellated by the shape labelled S in the above diagram. This shape is made up of one tile in the shape of a regular hexagon and two tiles in the shape of equilateral triangles.

The diagram shows that, as the side length of the triangular tiles is half that of the hexagonal tiles, the area of the hexagonal tiles is 24 times the area of the triangular tiles.

[This may be seen by counting the number of small triangles, each congruent to the triangular tiles, into which the hexagonal tile is divided by the dashed lines.

Alternatively, we see that the hexagonal tile may be divided into 6 equilateral triangles each with side length twice that of the triangular tiles, and hence each with an area 4 times that of the triangular tiles. It follows that the area of the hexagonal tile is 6×4, that is, 24, times that of the triangular tiles.]

Therefore the shape S is made up of 2 black triangles and a hexagon equal in area to 24 of the black triangles. So the total area of S is equal to that of $2 + 24 = 26$ of the triangles.

It follows that the fraction of the area of S that is black is $\frac{2}{26} = \frac{1}{13}$.

A large area of the floor will be covered by a large number of tiles which make up shapes congruent to S, together with a few bits of tiles. Therefore the fraction of the floor that is shaded black will be very close to $\frac{1}{13}$.