UK INTERMEDIATE MATHEMATICAL CHALLENGE

THURSDAY 5TH FEBRUARY 2015

Organised by the **United Kingdom Mathematics Trust**
from the School of Mathematics, University of Leeds

SOLUTIONS LEAFLET

This solutions leaflet for the IMC is sent in the hope that it might provide all concerned with some alternative solutions to the ones they have obtained. It is not intended to be definitive. The organisers would be very pleased to receive alternatives created by candidates.

For reasons of space, these solutions are necessarily brief. Extended solutions, and some exercises for further investigation, can be found at:

http://www.ukmt.org.uk/

The UKMT is a registered charity

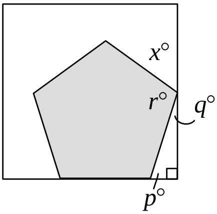
---

1. A $1 - 0.2 + 0.03 - 0.004 = 0.8 + 0.026 = 0.826.$

2. E The number of steps climbed per minute $\sim \frac{1600}{12} = \frac{400}{3} \sim 130.$

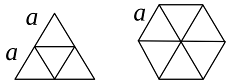
3. E Half of a third, plus a third of a quarter, plus a quarter of a fifth equals $\frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{10 + 5 + 3}{60} = \frac{18}{60} = \frac{3}{10}.$

4. C The sum of the exterior angles of a convex polygon equals 360°. The angle marked $p°$ is the exterior angle of a regular pentagon. So $p = 360 \div 5 = 72$. The angle sum of a triangle equals 180°, so $q = 180 - 90 - 72 = 18$. The angle marked $r°$ is the interior angle of a regular pentagon, so $r = 180 - 72 = 108$. The angles marked $q°$, $r°$ and $x°$ lie along a straight line, so $x = 180 - (q + r) = 180 - (18 + 108) = 54$.



5. B $1^6 = (1^3)^2$, $3^4 = (3^2)^2$, $4^3 = 64 = 8^2$ and 5^2 are all squares. However, $2^5 = 32$ and is not a square.

6. D Let the length of the side of the regular hexagon be a. Then its perimeter is $6a$. Therefore the perimeter of the equilateral triangle is also $6a$, so the length of each of its sides is $2a$. The diagrams show that the equilateral triangle may be divided up into 4 equilateral triangles of side a, whereas the regular hexagon may be divided into 6 such triangles. So the required ratio is 4 : 6 = 2 : 3.



7. E The tetrahedron has 6 edges and 4 vertices, so the required product is $6 \times 4 = 24$.

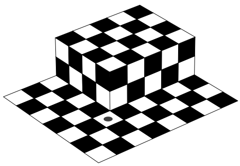
8. B The two-digit squares are 16, 25, 36, 49, 64 and 81. Of these, only 49 and 81 differ by 1 from a multiple of 10.

9. B The sum of the exterior angles of a convex polygon equals 360°. Therefore $p + r + t + v + x = 360°$. Similarly, $q + s + u + w + y = 360°$. Therefore $p + q + r + s + t + u + v + w + x + y = 720°$.

10. C $2^2 \times 3^3 \times 5^5 \times 7^7$ is of the form $2^2 \times$ an odd number. It therefore has the form $4(2n + 1) = 8n + 4$ where n is a positive integer and so leaves a remainder of 4 when divided by 8.

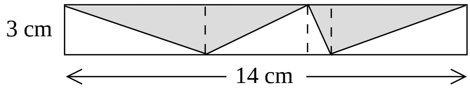
11. D As the 3 numbers have mean 7, their sum equals $3 \times 7 = 21$. For one of the numbers to be as large as possible the other two numbers must be as small as possible. They must also be different and so must be 1 and 2. Hence the largest possible of the three numbers equals $21 - (1 + 2) = 18$.

12. D If the ant moves alternately from white square to black square and from black to white, then it will end on a white square after 4 moves. So it must find a way to move from white to white or from black to black. However, there is only one pair of adjacent black squares and only one of white. To reach that pair of black squares, the ant must move to one side then climb up to one of the pair. That uses up 3 moves, and the fourth must take it to the other black square of that pair. Thus the two black squares in that pair are possible end points.



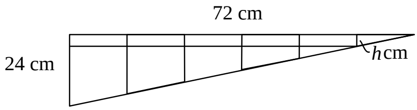
If, instead, the ant uses the white pair, it must first move to one side, then climb up to one of the white pair then across to the other square of that pair. That uses 3 moves. The fourth move can then take it to any of the three adjoining black squares. This gives 6 end squares, but these include the two already identified. So there are just 6 possible end squares which are black.

13. A Three vertical lines have been added to the diagram. These divide the original diagram into 4 rectangles. In each of these, a diagonal divides the rectangle into two triangles of equal area, one shaded and one unshaded. So the total shaded area in the original rectangle is equal to the total unshaded area and is therefore equal to half the area of the original rectangle. So the total shaded area is $\frac{1}{2} \times 3 \times 14\,\text{cm}^2 = 21\,\text{cm}^2$.



14. C Suppose the first three terms of the sequence are a, b, c. Then $c = \frac{1}{2}(a+b)$ and so $a + b = 2c$. The mean of the first three terms is then $\frac{1}{3}(a+b+c) = \frac{1}{3}(2c+c) = c$, so the fourth term is c. Similarly, the following terms are all equal to c. Since one of these terms is 26 and $a = 8$ then $b = 2c - a = 52 - 8 = 44$.

15. C The stripes are of equal width, so the width of each stripe is $(72 \div 6)\,\text{cm} = 12\,\text{cm}$. The diagram shows that the difference between the areas of any two adjacent stripes is equal to the area of a rectangle of width 12 cm and height h cm. By similar triangles, $\frac{h}{12} = \frac{24}{72}$. So $h = \frac{12 \times 24}{72} = 4$. So the required area is $12 \times 4\ \text{cm}^2 = 48\ \text{cm}^2$.

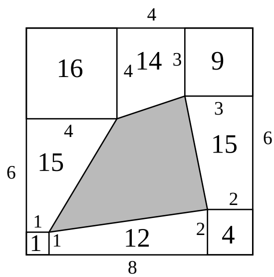


16. D All four values cannot be prime. If this were so, both $m \times n$ and $m \div n$ would be prime which can happen only if m is prime and $n = 1$. If m is an odd prime then $m + 1$ is even and at least 4, hence not prime, while if $m = 2$ then $m - 1$ is not prime but $m + 1 = 3$ is. Thus three prime values are the most we can have.

17. D The 12 pentagonal panels have a total of $12 \times 5 = 60$ edges. The 20 hexagonal panels have a total of $20 \times 6 = 120$ edges. So in total the panels have 180 edges. When the panels are sewn together, two edges form each join. So the number of joins is $180 \div 2 = 90$.

18. A Let the weights in kg of the box, 1 plate and 1 cup be b, p and c respectively.
Then: $b + 20p + 30c = 4.8$ (i); $b + 40p + 50c = 8.4$ (ii).
Subtracting (i) from (ii): $20p + 20c = 3.6$ (iii). So $10p + 10c = 1.8$ (iv).
Subtracting (iv) from (i): $b + 10p + 20c = 3$. So the required weight is 3 kg.

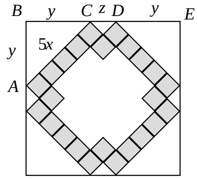
19. A The small numbers in the figure show the lengths in cm of each line segment. The larger numbers inside the figure show the areas in cm^2 of each square or trapezium. (The area of a trapezium is $\frac{1}{2}(a + b)h$ where a and b are the lengths of the parallel sides and h is the perpendicular distance between them.)
So the area of the shaded portion in cm^2 is
$11 \times 11 - (1 + 12 + 4 + 15 + 9 + 14 + 16 + 15) = 35$.
(*See the extended solutions for a beautifully elegant solution of this problem*.)



20. C There are 3 different possibilities for the first character. The second character may be any digit from 0 to 9 inclusive, so it has 10 different possibilities. The third character differs from the second digit, so has 9 different possibilities. Once the second and third characters are determined, the fourth character is also determined since it is the units digit of the sum of the second and third characters.
So, the number of different codes is $3 \times 10 \times 9 = 270$.

21. D Let the area of each rectangle be Y. Then the total shaded area is $2(Y - X) + X = 2Y - X$. Therefore $X = \frac{1}{8}(2Y - X)$. So $8X = 2Y - X$, that is $9X = 2Y$. Therefore $\frac{X}{Y} = \frac{2}{9}$.

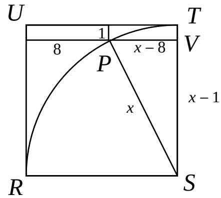
22. B Let the length of the sides of each small square be x. Then the shaded area is $24x^2$. Let the perimeter of the square be divided into eight line segments, each of length y, and four line segments of length z. Some of these are labelled in the diagram. By Pythagoras' Theorem in triangle ABC:
$y^2 + y^2 = (5x)^2$, that is $2y^2 = 25x^2$.



So $y = \frac{5}{\sqrt{2}}x = \frac{5\sqrt{2}}{2}x$. Similarly, in the triangle with hypotenuse CD:
$x^2 + x^2 = z^2$, that is $2x^2 = z^2$. So $z = \sqrt{2}x$. Therefore the length of the side of the large square is $2y + z = 5\sqrt{2}x + \sqrt{2}x = 6\sqrt{2}x$. So the area of the large square is $\left(6\sqrt{2}x\right)^2 = 72x^2$. Hence the required fraction is $\frac{24x^2}{72x^2} = \frac{1}{3}$.

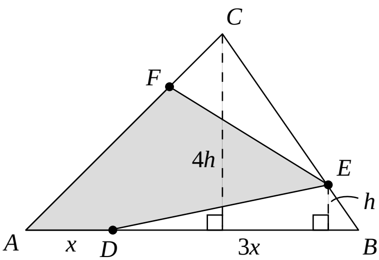
23. B The permutations which follow UKIMC in dictionary order are UKMCI, UKMIC, UMCIK, UMCKI, UMICK, UMIKC, UMKCI, UMKIC. There are eight of these, so UKIMC is 112th in the list.

24. E In the diagram V is the point where the perpendicular from P meets TS. Let the side of the square $RSTU$ be x. So the radius of the arc from R to T is x. Therefore SP has length x, PV has length $x - 8$ and VS has length $x - 1$. Applying Pythagoras' Theorem to triangle PVS:
$(x - 8)^2 + (x - 1)^2 = x^2$. So $x^2 - 16x + 64 + x^2 - 2x + 1 = x^2$.
Therefore $x^2 - 18x + 65 = 0$, so $(x - 5)(x - 13) = 0$.



Hence $x = 5$ or $x = 13$, but $x > 8$ so the length of the side of the square $RSTU$ is 13.

25. D Points A, B, C, D, E, F on the perimeter of the triangle are as shown. Let AD have length x so that DB has length $3x$. Let the perpendicular from C to AB have length $4h$. So, by similar triangles, the perpendicular from E to DB has length h. The area of triangle ABC is $\frac{1}{2} \times 4x \times 4h = 8xh$. The area of triangle DBE is $\frac{1}{2} \times 3x \times h = \frac{3}{2}xh$. So the area of triangle DBE is $\frac{3}{16}$ of the area of triangle ABC.



Similarly, by drawing perpendiculars to CB from A and from F, it may be shown that the area of triangle FEC is $\frac{3}{16}$ of the area of triangle ABC.
So the fraction of the area of the triangle that is shaded is $1 - \frac{3}{16} - \frac{3}{16} = \frac{5}{8}$.