UK INTERMEDIATE MATHEMATICAL CHALLENGE

February 2nd 2012

EXTENDED SOLUTIONS

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some ***Extension Problems*** for further investigations.

The Intermediate Mathematical Challenge (IMC) is a multiple choice contest, in which you are presented with five alternative answers, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the IMC, and we often give first a solution using this approach.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. So usually we have included a complete solution which does not use the fact that one of the given alternatives is correct. Thus we have aimed to give full solutions with all steps explained. We therefore hope that these solutions can be used as a model for the type of written solution that is expected in the Intermediate Mathematical Olympiad and similar competitions.

We welcome comments on these solutions, and, especially, corrections or suggestions for improving them. Please send your comments,

either by e-mail to

enquiry@ukmt.co.uk

or by post to

IMC Solutions, UKMT Maths Challenges Office, School of Mathematics,
University of Leeds, Leeds LS2 9JT.

Quick Marking Guide

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
B	D	E	C	D	A	C	A	B	D	C	B	D	A	E	B	E	C	A	D	A	E	C	D	B

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1.	How many of the following four numbers are prime?				
	3	33	333	3333	
	A 0	B 1	C 2	D 3	E 4

Solution: **B**

The number 3 is prime, but the other numbers listed are not prime as $33 = 3 \times 11$, $333 = 3 \times 111$ and $3333 = 3 \times 1111$.

Extension problems

In general, a positive integer whose digits are all 3s is divisible by 3, since

$$333...333 = 3 \times 111...111.$$

Hence, except for the number 3 itself, such an integer is not prime. A similar remark applies if 3 is replaced by any of the digits 2, 4, 5, 6, 7, 8 and 9 (except that in the cases of the digits 4, 6, 8 and 9, the number consisting of a single digit is also not prime). This leaves the case of numbers all of whose digits are 1s. This case is considered in the following problems.

1.1 Check which of the numbers 1, 11, 111 and 1111, if any, are prime.

1.2 Show that a positive integer all of whose digits are 1s, and which has an even number of digits, is not prime.

1.3 Show that a positive integer all of whose digits are 1s, and which has a number of digits which is a multiple of 3, is not prime.

1.4 Show that a positive integer all of whose digits are 1s, and which has a number of digits which is not a prime number, is itself not a prime number.

1.5 It follows from 1.4 that a number all of whose digits are 1s can be prime only if it has a prime number of these digits. However numbers of this form need not be prime. Thus 11 with 2 digits is prime, but 111 with 3 digits is not. Determine whether 11111, with 5 digits, is prime.

Without a computer quite a lot of arithmetic is needed to answer question 1.5. As numbers get larger it becomes more and more impractical to test by hand whether they are prime. Using a computer we can see that 1111111, 11111111111, 1111111111111 and 11111111111111111 with 7, 11, 13 and 17 digits, respectively, are not prime. In fact, $1111111 = 239 \times 4649$, $11111111111 = 21649 \times 513239$, $1111111111111 = 53 \times 79 \times 265371653$ and $11111111111111111 = 2071723 \times 5363222357$, where the given factors are all prime numbers.

The next largest example after 11 of a number all of whose digits are 1s and which is prime is 1111111111111111111 with 19 digits. The next largest has 23 digits, and the next largest after that has 317 digits. It is not known whether there are infinitely many prime numbers of this form. We have taken this information from the book *The Penguin Dictionary of Curious and Interesting Numbers* by David Wells, Penguin Books, 1986.

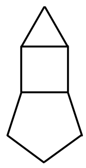
2. Three positive integers are all different. Their sum is 7. What is their product?

A 12 B 10 C 9 D 8 E 5

Solution: **D**

It can be seen that $1+2+4=7$ and $1\times 2\times 4=8$, so assuming that there is just one solution, the answer must be 8. In the context of the IMC, that is enough, but if you are asked to give a full solution, you need to give an argument to show there are no other possibilities. This is not difficult. For suppose a, b and c are three different positive integers with sum 7, and that $a<b<c$. If $a\geq 2$, then $b\geq 3$ and $c\geq 4$, and so $a+b+c\geq 9$. So we must have that $a=1$. It follows that $b+c=6$. If $b\geq 3$ then $c\geq 4$ and hence $b+c\geq 3+4=7$. So $b=2$. Since $a=1$ and $b=2$, it follows that $c=4$.

3. An equilateral triangle, a square and a pentagon all have the same side length. The triangle is drawn on and above the top edge of the square and the pentagon is drawn on and below the bottom edge of the square. What is the sum of the interior angles of the resulting polygon?

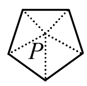


A 10×180^0 B 9×180^0 C 8×180^0 D 7×180^0 E 6×180^0

Solution: **E**

The sum of the interior angles of the polygon is the sum of the angles in the triangle, the square and the pentagon. The sum of the interior angles of the triangle is 180^0, and the sum of the angles of the square is $360^0=2\times 180^0$, and the sum of the angles of the pentagon is $540^0=3\times 180^0$.

So the sum of the angles is $(1+2+3)\times 180^0=6\times 180^0$.



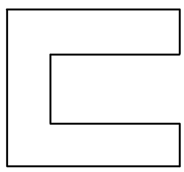
Note: There is more than one way to see that the sum of the angles of a pentagon is 540^0. Here is one method. Join the vertices of the pentagon to some point, say P, inside the pentagon. This creates 5 triangles whose angles sum to 5×180^0.The sum of the angles in these triangles is the sum of the angles in a pentagon plus the sum of the angles at P, which is $360^0=2\times 180^0$. So the sum of the angles in the pentagon is $5\times 180^0-2\times 180^0=3\times 180^0$.

Extension Problems

3.1 What is the sum of the angles in a septagon?

3.2 What is the sum of the angles in a polygon with n vertices?

3.3 Does your method in 3.2 apply to a polygon shaped as the one shown where you cannot join all the vertices by straight lines to a point inside the polygon? If not, how could you modify your method to cover this case?



4. All four digits of two 2-digit numbers are different. What is the largest possible sum of two such numbers?

A 169 B 174 C 183 D 190 E 197

Solution: **C**

To get the largest possible sum we need to take 9 and 8 as the tens digits, and 7 and 6 as the units digits. For example,

$$\begin{array}{ccc} & 9 & 7 \\ + & 8 & 6 \\ \hline 1 & 8 & 3 \\ \hline \end{array}$$

Extension Problem

4.1 All nine digits of three 3-digit numbers are different. What is the largest possible sum of three such numbers?

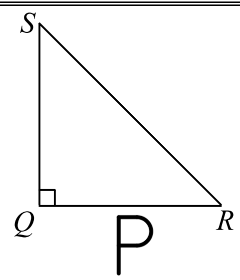
5. How many minutes will elapse between 20:12 today and 21:02 tomorrow?

A 50 B 770 C 1250 D 1490 E 2450

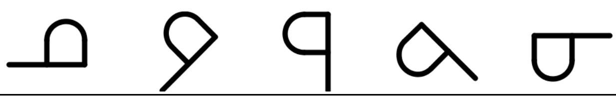
Solution: **D**

From 20:12 today until 20.12 tomorrow is 24 hours, that is $24 \times 60 = 1440$ minutes. There are 50 minutes from 20:12 tomorrow to 21:02 tomorrow. This gives a total of $1440 + 50 = 1490$ minutes.

6. Triangle *QRS* is isosceles and right-angled.
Beatrice reflects the P-shape in the side *QR* to get an image.
She reflects the first image in the side *QS* to get a second image.
Finally, she reflects the second image in the side *RS* to get a third image.
What does the third image look like?

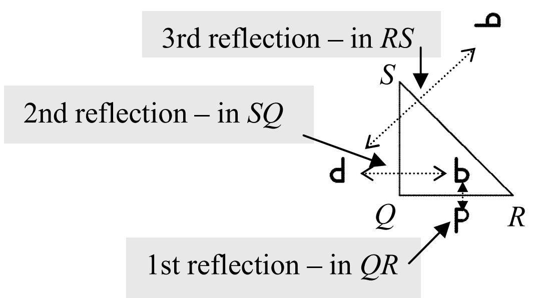


A B C D E



Solution: **A**

The effect of the successive reflections is shown in the diagram.

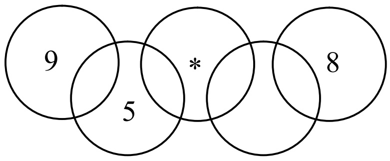


7.	The prime numbers p and q are the smallest primes that differ by 6. What is the sum of p and q?				
	A 12	B 14	C 16	D 20	E 28

Solution: **C**

Suppose $p < q$. Then $q = p + 6$. The prime numbers are 2, 3, 5, 7, …. . With $p = 2$, $q = 8$, which is not prime. Similarly if $p = 3$, $q = 9$, which is also not prime. However, when $p = 5$, $q = 11$, which is prime. So, $p = 5$, $q = 11$ gives the smallest primes that differ by 6. Then $p + q = 5 + 11 = 16$.

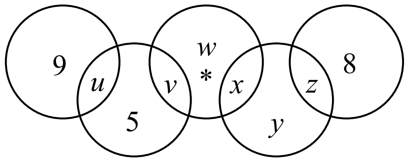
8. Seb has been challenged to place the numbers 1 to 9 inclusive in the nine regions formed by the Olympic rings so that there is exactly one number in each region and the sum of the numbers in each ring is 11. The diagram shows part of his solution.



What number goes in the region marked * ?

A 6 B 4 C 3 D 2 E 1

Solution: **A**



We let *u*, *v*, *w*, *x*, *y* and *z* be the numbers in the regions shown. Since the sum of the numbers in each ring is 11, we have, from the leftmost ring, that $9 + u = 11$ and so $u = 2$. Then, from the next ring, $2 + 5 + v = 11$ and so $v = 4$. From the rightmost ring, $z + 8 = 11$ and so $z = 3$.

We have now used the digits 2, 3, 4, 5, 8 and 9, leaving 1, 6 and 7 .

From the middle ring we have that $4 + w + x = 11$, and so $w + x = 7$. From the second ring from the right $x + y + 3 = 11$, and so $x + y = 8$. So we need to solve the equations $w + x = 7$ and $x + y = 8$, using 1, 6 and 7. It is easy to see that the only solution is $x = 1$, $y = 7$ and $w = 6$. So 6 goes in the region marked *.

9. Auntie Fi's dog Itchy has a million fleas. His anti-flea shampoo claims to leave no more than 1% of the original number of fleas after use. What is the least number of fleas that will be eradicated by the treatment?

A 900 000 B 990 000 C 999 000 D 999 990 E 999 999

Solution: **B**

Since no more than 1% of the fleas will remain, at least 99% of them will be eradicated. Now 99% of a million is

$$\frac{99}{100} \times 1\,000\,000 = 99 \times 10\,000 = 990\,000\,.$$

10. An ‘abundant’ number is a positive integer N, such that the sum of the factors of N (excluding N itself) is greater than N.. What is the smallest abundant number?

A 5 B 6 C 10 D 12 E 15

Solution: **D**

In the IMC, it is only necessary to check the factors of the numbers given as the options. However, to be sure that the smallest of these which is abundant, is the overall smallest abundant number, we would need to check the factors of all the positive integers in turn, until we find an abundant number. The following table gives the sum of the factors of N (excluding N itself), for $1 \leq N \leq 12$.

N	1	2	3	4	5	6	7	8	9	10	11	12
factors of N, excluding N	-	1	1	1,2	1	1,2,3	1	1,2,4	1,3	1,2,5	1	1,2,3,4,6
sum of these factors	0	1	1	3	1	6	1	7	4	8	1	16

From this table we see that 12 is the smallest abundant number.

Extension Problems

10.1. Which is the next smallest abundant number after 12?

10.2. Show that if n is a power of 2, and $n > 2$ (that is, $n =$ 4, 8, 16, .. etc) then $3n$ is an abundant number.

10.3 Prove that if n is an abundant number, then so too is each multiple of n.

10.4 A number, N, is said to be *deficient* if the sum of the divisors of N, excluding N itself, is less than N. Prove that if N is a power of 2, then N is a deficient number.

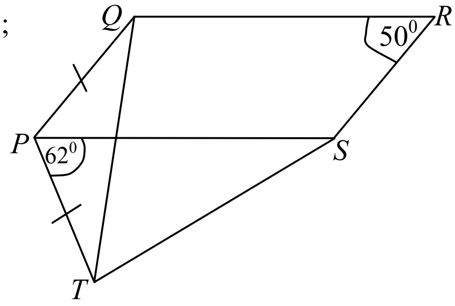
10.5 A number, N, is said to be *perfect* if the sum of the divisors of N, excluding N itself, is equal to N. We see from the above table that 6 is the smallest perfect number. Find the next smallest perfect number.

Note: It follows from Problems 10.2 and 10.4 that there are infinitely many abundant numbers and infinitely many deficient numbers. It remains an open question as to whether there are infinitely many perfect numbers. In Euclid’s *Elements* (Book IX, Proposition 36) it is proved that even integers of the form $2^{p-1}(2^p - 1)$, where $2^p - 1$ is a prime number are perfect (for example, the perfect number 6 corresponds to the case where $p = 2$). Euclid lived around 2300 years ago. It took almost 2000 years before the great Swiss mathematician Leonard Euler showed that, conversely, all even perfect numbers are of the form $2^{p-1}(2^p - 1)$, where $2^p - 1$ is prime. Euler lived from 1707 to 1783, but his theorem about perfect numbers was not published until 1849. It is still not known whether there are infinitely many even perfect numbers, as we don’t know whether there are infinitely many primes of the form $2^p - 1$. It is also not known whether there are any odd perfect numbers.

11. In the diagram, $PQRS$ is a parallelogram; $\angle QRS = 50^0$; $\angle SPT = 62^0$ and $PQ = PT$.

What is the size of $\angle TQR$?

A 84^0 B 90^0 C 96^0 D 112^0 E 124^0



Solution: **C**

Because $PQRS$ is a parallelogram, $\angle SPQ = \angle QRS = 50^0$. Therefore $\angle TPQ = (62+50)^0 = 112^0$. Therefore, as the angles in a triangle add up to 180^0, $\angle PQT + \angle PTQ = 180^0 - 112^0 = 68^0$. Because *PQ*=*PT*, the triangle *QPT* is isosceles, and so $\angle PQT = \angle PTQ$. Therefore $\angle PQT = \angle PTQ = 34^0$. Because $PQRS$ is a parallelogram, $\angle PQR + \angle QRS = 180^0$, and therefore $\angle PQR = 180^0 - 50^0 = 130^0$. Therefore, $\angle TQR = \angle PQR - \angle PQT = 130^0 - 34^0 = 96^0$.

12. Which of the following has a different value from the others?

A 18% of £30 B 12% of £50 C 6% of £90 D 4% of £135 E 2% of £270

Solution: **B**

We have that 18% of £30 $= £(\frac{18}{100} \times 30) = £5.40$. Similarly, 12% of £50 is £6.00, and 6% of £90 is £5.40. We already see that option B must be the odd one out. It is easy to check that 4% of £135 and 2% of £270 are also both £5.40.

13. Alex Erlich and Paneth Farcas shared an opening rally of 2 hours and 12 minutes during their table tennis match at the 1936 World Games. Each player hit around 45 shots per minute. Which of the following is closest to the total number of shots played in the rally?

A 200 B 2000 C 8000 D 12 000 E 20 000

Solution: **D**

Since they each hit about 45 shots in one minute, between them they hit about 90 shots per minute. Now 2 hours and 12 minutes is 132 minutes. So the total number of shots in the match is 90×132, and 90×132 is approximately $100 \times 120 = 12\,000$.

Extension Problem

13.1 Note that 90 is 90% of 100 and 132 is 110% of 120. What is the percentage error in approximating 90×132 by 100×120?

14. What value of x makes the mean of the first three numbers in this list equal to the mean of the last four?

15 5 x 7 9 17

A 19 B 21 C 24 D 25 E 27

Solution: **A**

The mean of the first three numbers in the list is $\frac{1}{3}(15+5+x)$ and the mean of the last four is $\frac{1}{4}(x+7+9+17)$. Now,

$$\frac{1}{3}(15+5+x)=\frac{1}{4}(x+7+9+17) \Leftrightarrow 4(15+5+x)=3(x+7+9+17)$$
$$\Leftrightarrow 80+4x=3x+99$$
$$\Leftrightarrow x=19.$$

An alternative method in the context of the IMC would be just to try the given options in turn. This runs the risk of involving a lot of arithmetic, but here, as the first option is the correct answer, the gamble would pay off.

15. Which of the following has a value that is closest to 0?

A $\frac{1}{2}+\frac{1}{3}\times\frac{1}{4}$ B $\frac{1}{2}+\frac{1}{3}\div\frac{1}{4}$ C $\frac{1}{2}\times\frac{1}{3}\div\frac{1}{4}$ D $\frac{1}{2}-\frac{1}{3}\div\frac{1}{4}$ E $\frac{1}{2}-\frac{1}{3}\times\frac{1}{4}$

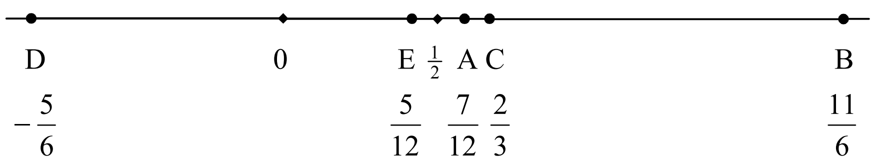
Solution: **E**

When working out the values of these expressions it is important to remember the convention (sometimes known as BODMAS or BIDMAS) that tells us that Divisions and Multiplications are carried out before Additions and Subtractions.

Some work can be saved by noting that the expressions A and B have values greater than $\frac{1}{2}$, whereas the value of expression E lies between 0 and $\frac{1}{2}$. So it must be C, D or E that has the value closest to 0.

Now, noting that $\frac{1}{3}\div\frac{1}{4}=\frac{1}{3}\times\frac{4}{1}=\frac{4}{3}$, we obtain that the value of C is $\frac{1}{2}\times\frac{1}{3}\div\frac{1}{4}=\frac{1}{2}\times\frac{4}{3}=\frac{2}{3}$; that of D is $\frac{1}{2}-\frac{1}{3}\div\frac{1}{4}=\frac{1}{2}-\frac{4}{3}=-\frac{5}{6}$; and that of E is $\frac{1}{2}-\frac{1}{3}\times\frac{1}{4}=\frac{1}{2}-\frac{1}{12}=\frac{5}{12}$.

From these calculations we see that E gives the value closest to 0.

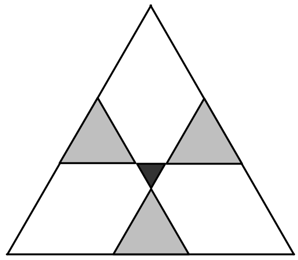


[The value of A is $\frac{1}{2}+\frac{1}{3}\times\frac{1}{4}=\frac{1}{2}+\frac{1}{12}=\frac{7}{12}$; and that of B is B $\frac{1}{2}+\frac{1}{3}\div\frac{1}{4}=\frac{1}{2}+\frac{4}{3}=\frac{11}{6}$.]

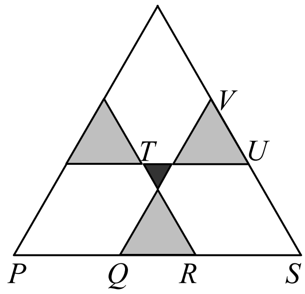
16. The diagram shows a large equilateral triangle divided by three straight lines into seven regions. The three grey regions are equilateral triangles with sides of length 5 cm and the central black region is an equilateral triangle with sides of length 2 cm.

What is the side length of the original large triangle?

A 18 cm B 19 cm C 20 cm D 21 cm E 22cm



Solution : **B**



Let *P, Q, R, S, T, U* and *V* be the points shown. All the angles in all the triangles are 60^0. So $\angle QRT = \angle PSU$ and hence *RT* is parallel to *SU*. Similarly, as $\angle RSV = \angle TUV$, *RS* is parallel to *TU*. Therefore *RSUT* is a parallelogram. Therefore *RS* has the same length as *TU*, namely, $2+5=7$ cm. Similarly *PQ* has length 7 cm. So the length of *PS* which is the sum of the lengths of *PQ, QS* and RS is $7+5+7=19$ cm.

17. The first term in a sequence of positive integers is 6. The other terms in the sequence follow these rules:

if a term is even then divide it by 2 to obtain the next term;
if a term is odd then multiply it by 5 and subtract 1 to obtain the next term.

For which values of n is the n th term equal to n?

A 10 only B 13 only C 16 only D 10 and 13 only E 13 and 16 only

Solution: **E**

Since the options refer only to the 10th, 13th and 16th terms of the sequence, as far as this IMC question is concerned it is only necessary to check the first 16 terms in the sequence. These are as shown in the table below:

n	n th term	=		n	n th term	=
1		6		9	$42 \div 2$	21
2	$6 \div 2$	3		10	$21 \times 5 - 1$	104
3	$3 \times 5 - 1$	14		11	$104 \div 2$	52
4	$14 \div 2$	7		12	$52 \div 2$	26
5	$7 \times 5 - 1$	34		**13**	$26 \div 2$	**13**
6	$34 \div 2$	17		14	$13 \times 5 - 1$	64
7	$17 \times 5 - 1$	84		15	$64 \div 2$	32
8	$84 \div 2$	42		**16**	$32 \div 2$	**16**

From this we see that the 13th term is 13, and the 16th term is 16, and that these are the only cases where the n th term is equal to n.

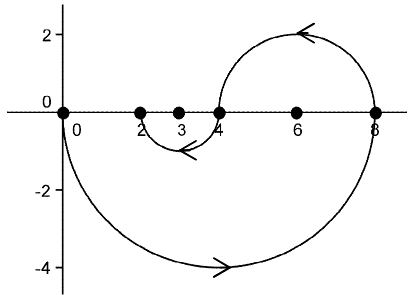
However, a complete answer requires a proof that for all $n > 16$, the nth term is not equal to n. It can be seen that after the 16th term the sequence continues 8, 4, 2, 1, 4, 2, 1… with the cycle 4, 2, 1 now repeating for ever. It follows that, for $n \geq 17$, the only values taken by the nth term are 8, 4, 2 and 1. We deduce that for $n > 16$, the nth term is not equal to n.

18. Peri the winkle starts at the origin and slithers anticlockwise around a semicircle with centre $(4, 0)$. Peri then slides anticlockwise around a second semicircle with centre $(6, 0)$, and finally clockwise around a third semicircle with centre $(3, 0)$.

Where does Peri end this expedition?

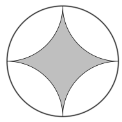
A $(0, 0)$ B $(1, 0)$ C $(2, 0)$ D $(4, 0)$ E $(6, 0)$

Solution: **C**



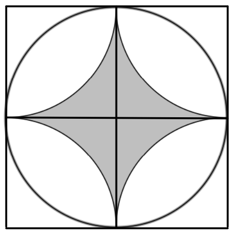
As may be seen from the diagram, Peri first moves along the semicircle with centre $(4, 0)$ from the point $(0, 0)$ to the point $(8, 0)$, then along the semicircle with centre $(6, 0)$ to the point $(4, 0)$, and finally along the semicircle with centre $(3, 0)$ to end up at the point $(2, 0)$.

19. The shaded region shown in the diagram is bounded by four arcs, each of the same radius as that of the surrounding circle. What fraction of the surrounding circle is shaded?



A $\frac{4}{\pi}-1$ B $1-\frac{\pi}{4}$ C $\frac{1}{2}$ D $\frac{1}{3}$ E it depends on the radius of the circle

Solution: **A**



Suppose that the surrounding circle has radius *r*. In the diagram we have drawn the square with side length 2*r* which touches the circle at the points where it meets the arcs. The square has area $(2r)^2 = 4r^2$. The unshaded area inside the square is made up of four quarter circles with radius *r*, and thus has area πr^2. Hence the shaded area is $4r^2 - \pi r^2 = (4-\pi)r^2$. The circle has area πr^2. So the fraction of the circle that is shaded is

$$\frac{(4-\pi)r^2}{\pi r^2} = \frac{4-\pi}{\pi} = \frac{4}{\pi} - 1.$$

20. A rectangle with area 125 cm^2 has sides in the ratio 4:5. What is the perimeter of the rectangle?

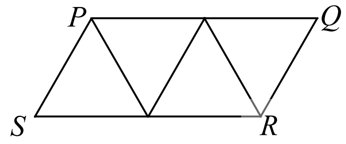
A 18 cm B 22.5 cm C 36 cm D 45 cm E 54 cm

Solution: **D**

Since the side lengths of the rectangle are in the ratio 4:5, they are 4*a* cm and 5*a* cm, for some positive number *a*. This means that the rectangle has area $4a \times 5a = 20a^2$ cm^2. Hence $20a^2 = 125$. So $a^2 = \frac{125}{20} = \frac{25}{4}$, and hence $a = \frac{5}{2}$. Hence the rectangle has perimeter $2(4a+5a) = 18a = 18 \times \frac{5}{2} = 45$ cm.

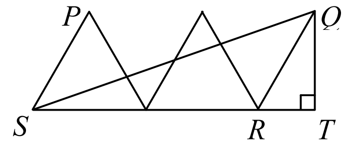
21. The parallelogram $PQRS$ is formed by joining together four equilateral triangles of side 1 unit, as shown.

What is the length of the diagonal SQ?



A $\sqrt{7}$ B $\sqrt{8}$ C 3 D $\sqrt{6}$ E $\sqrt{5}$

Solution: **A**



Let T be the foot of the perpendicular from Q to the line SR extended. Now RQT is half of an equilateral triangle with side length 1. Hence the length of RT is $\frac{1}{2}$ and hence ST has length $1+1+\frac{1}{2}=\frac{5}{2}$. By Pythagoras' Theorem applied to the right angled triangle RQT, $(1)^2=(\frac{1}{2})^2+QT^2$. Therefore $QT^2=(1)^2-(\frac{1}{2})^2=1-\frac{1}{4}=\frac{3}{4}$. Hence, by Pythagoras' Theorem applied to the right angled triangle SQT, $SQ^2=ST^2+QT^2=(\frac{5}{2})^2+\frac{3}{4}=\frac{25}{4}+\frac{3}{4}=7$. Therefore, $SQ=\sqrt{7}$.

22. What is the maximum possible value of the median number of cups of coffee bought per customer on a day when Sundollars Coffee Shop sells 477 cups of coffee to 190 customers, and every customer buys at least one cup of coffee?

A 1.5 B 2 C 2.5 D 3 E 3.5

Solution: **E**

Put the set of numbers of cups of coffee drunk by the individual customers into numerical order with the smallest first. This gives an increasing sequence of positive integers with sum 477. Because 190 is even, the median of these numbers is the mean of the 95th and 96th numbers in this list. Suppose these are a and b, respectively. Then the median is $\frac{1}{2}(a+b)$.

We note that $1 \le a \le b$. Also, each of the first 94 numbers in the list is between 1 and a, and each of the last 94 numbers is at least b. So if we replace the first 94 numbers by 1, and the last 94 numbers by b, we obtain the sequence of numbers

$$\underbrace{1,1,1,\ldots,1}_{94},a,\underbrace{b,b,b,\ldots\ldots,b}_{95} \qquad (1)$$

whose sum does not exceed 477, the sum of the original sequence. Therefore

$$94+a+95b \le 477 \qquad (2)$$

As $1 \le a$, it follows that $95+95b \le 94+a+95b \le 477$, hence $95b \le 477-95=382$ and therefore

$b \le \frac{382}{95}$. Therefore, since b is an integer, $b \le 4$.

When $b = 4$, it follows from (2) that $94 + a + 380 \le 477$, giving $a \le 3$.

This shows that the maximum possible values for a and b are 3 and 4, respectively. We can see that these values are possible, as, if we substitute these values in (1), we obtain a sequence of numbers with sum $94 \times 1 + 3 + 95 \times 4 = 477$. So 3.5 is the maximum possible value of the median.

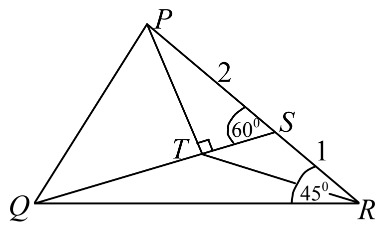
Extension Problem

22.1 What is the maximum possible value of the median number of cups of coffee bought per customer on a day when the Sundollars Coffee Shop sells 201 cups of coffee to 100 customers, and every customer buys at least one cup of coffee?

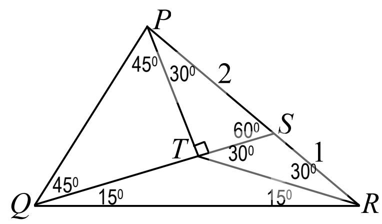
23. In the triangle PQR, $PS = 2$; $SR = 1$; $\angle PRQ = 45^0$; T is the foot of the perpendicular from P to QS and $\angle PST = 60^0$.

What is the size of $\angle QPR$?

A 45^0 B 60^0 C 75^0 D 90^0 E 105^0



Solution: **C**



In the triangle PST, $\angle PTS = 90^0$ and $\angle PST = 60^0$. Therefore $\angle TPS = 30^0$ and the triangle PST is half of an equilateral triangle. It follows that $ST = \frac{1}{2}PS = 1$. Therefore triangle RST is isosceles, and hence $\angle STR = \angle SRT$. By the Exterior Angle Theorem, $\angle PST = \angle STR + \angle SRT$. Therefore $\angle STR = \angle SRT = 30^0$. Hence $\angle QRT = \angle PRQ - \angle SRT = 45^0 - 30^0 = 15^0$. Using the Exterior Angle Theorem again, it follows that $\angle STR = \angle TQR + \angle QRT$, and hence $\angle TQR = \angle STR - \angle QRT = 45^0 - 30^0 = 15^0$. Therefore the base angles of triangle TQR are equal. Hence TQR is an isosceles triangle, and so $QT = RT$. We also have that the base angles in triangle TPR are both equal to 30^0, and so $PT = RT$. Therefore $QT = RT = PT$. So PTQ is an isosceles right-angled triangle. Therefore $\angle QPT = 45^0$. Finally, we deduce that $\angle QPR = \angle QPT + \angle TPS = 45^0 + 30^0 = 75^0$.

24. All the positive integers are written in the cells of a square grid. Starting from 1, the numbers spiral anticlockwise. The first part of the spiral is shown in the diagram.

Which number is immediately below 2012?

A 1837 B 2011 C 2013 D 2195 E 2210

					...	32	31	
		17	16	15	14	13	30	
		18	5	4	3	12	29	
		19	6	1	2	11	28	
		20	7	8	9	10	27	
		21	22	23	24	25	26	

Solution: **D**

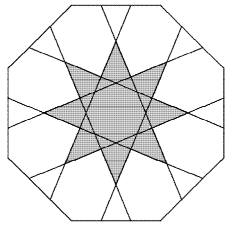
The key to the solution is to note that the squares of the odd numbers occur on the diagonal leading downwards and to the right from the cell which contains the number 1, and the squares of the even numbers occur on the diagonal which leads upwards and to the left of the cell which contains the number 4.

The squares of the even numbers have the form $(2n)^2$, that is, $4n^2$. We see that the number $4n^2+1$ occurs to the left of the cell containing $4n^2$. Below $4n^2+1$ there occur the numbers $4n^2+2, 4n^2+3, .., 4n^2+2n+1$, and then in the cells to the right of the cell containing $4n^2+2n+1$, there occur the numbers $4n^2+2n+2, 4n^2+2n+3, ..., 4n^2+4n+1=(2n+1)^2$.

Now $44^2=1936$ and $45^2=2025$. Thus 2011 is in the same row as 2025 and to the left of it, in the sequence 1981,….,2012,…,2025, and below these occur the numbers $2163, \ldots, 2209=47^2$, with 2208 below 2025 as shown below. It follows that 2195 is the number below 2012. In the diagram below the square numbers are shown in bold.

	1937	**1936**	...	...	...	...	...	...	...	...	...	...	...	...	...	
	⋮															
	⋮						**...**	32	31							
	⋮			17	**16**	15	14	13	30							
	⋮			18	5	**4**	3	12	29							
	⋮			19	6	**1**	2	11	28							
	⋮			20	7	8	**9**	10	27							
	⋮			21	22	23	24	**25**	26							
	⋮															
	1981	1982	...	...	...	...	...	...	...	...	2012	...	2024	**2025**	...	
	2164	2165	...	...	...	...	...	...	...	...	2195	...	2207	2208	**2209**	

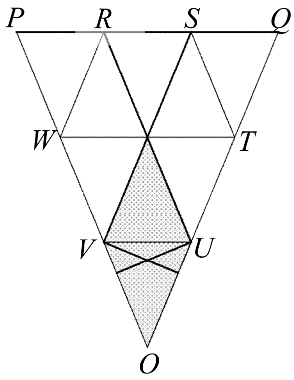
25. The diagram shows a ceramic design by the Catalan architect Antoni Gaudi. It is formed by drawing eight lines connecting points which divide the edges of the outer regular octagon into three equal parts, as shown.

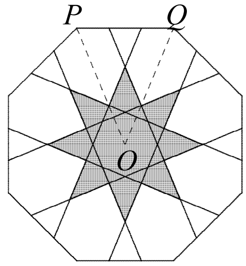


What fraction of the octagon is shaded?

A $\frac{1}{5}$ B $\frac{2}{9}$ C $\frac{1}{4}$ D $\frac{3}{10}$ E $\frac{5}{16}$

Solution: **B**





We consider the triangular segment of the octagon formed by joining two adjacent vertices, P and Q to the centre, O. For convenience, we show this segment, drawn on a larger scale, on the left, where we have added the lines RW, ST, TW and UV. These lines are parallel to the edges of the triangle POQ, as shown and together with the lines RU and SV they divide the triangle OPQ into 9 congruent triangles, of which 2 are shaded. Thus $\frac{2}{9}$ of the segment is shaded. The same holds for all the other congruent segments of the octagon. So $\frac{2}{9}$ of the whole octagon is shaded.

Extension Problem

25.1 In the solution we have said that the triangle OPQ is divided into 9 congruent triangles, but we have not justified the claim that the triangles are congruent. Complete the argument by giving a proof that these triangles are congruent.

UK INTERMEDIATE MATHEMATICAL CHALLENGE

THURSDAY 2nd FEBRUARY 2012

Organised by the **United Kingdom Mathematics Trust from the School of Mathematics, University of Leeds**

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SOLUTIONS LEAFLET

This solutions leaflet for the IMC is sent in the hope that it might provide all concerned with some alternative solutions to the ones they have obtained. It is not intended to be definitive. The organisers would be very pleased to receive alternatives created by candidates.

The UKMT is a registered charity

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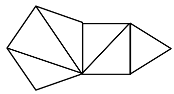
1. B 3 is the only one of the four numbers which is prime. The sums of the digits of the other three numbers are 6, 9, 12 respectively. These are all multiples of 3, so 33, 333, 3333 are all multiples of 3.

2. D The following triples of positive integers all sum to 7:

(1, 1, 5), (1, 2, 4), (1, 3, 3); (2, 2, 3).

In only one of these are the three integers all different, so the required integers are 1, 2, 4 and their product is 8.

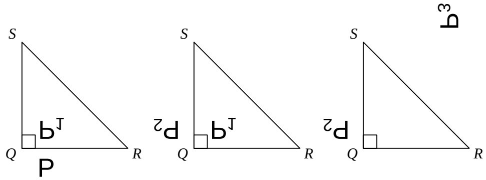
3. E The diagram shows that the interior angles of the polygon may be divided up to form the interior angles of six triangles. So their sum is $6 \times 180°$.



4. **C** The digits to be used must be 9, 8, 7, 6. If any of these were to be replaced by a smaller digit, then the sum of the two two-digit numbers would be reduced. For this sum to be as large as possible, 9 and 8 must appear in the 'tens' column rather than the 'units' column. So the largest possible sum is 97 + 86 or 96 + 87. In both cases the sum is 183.

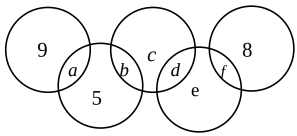
5. **D** The difference between the two given times is 24 hours 50 minutes = (24 × 60 + 50) minutes = (1440 + 50) minutes = 1490 minutes.

6. **A** The diagram shows the result of the successive reflections.



7. **C** The primes in question are 5 and 11. The only primes smaller than 5 are 2 and 3. However neither 8 nor 9 is prime so *p* and *q* cannot be 2 and 8, nor 3 and 9.

8. **A** Referring to the diagram, $a = 11 - 9 = 2$; $b = 11 - 5 - a = 4$; $f = 11 - 8 = 3$. So the values of *c*, *d* and *e* are 1, 6, 7 in some order. If $c = 1$ then $d = 6$, but then *e* would need to be 2, not 7. If $c = 6$, then $d = 1$ and $e = 7$ and this is a valid solution. Finally, if $c = 7$ then *d* would need to equal 0, which is not possible. So in the only possible solution, * is replaced by 6.



9. **B** 1% of 1 000 000 = 1 000 000 ÷ 100 = 10 000. So the least number of fleas which will be eradicated is 1 000 000 − 10 000 = 990 000.

10. **D** The table shows the first 12 positive integers, *N*, and the sum, *S*, of the factors of *N* excluding *N* itself. As can be seen, 12 is the first value of *N* for which this sum exceeds *N*, so 12 is the smallest abundant number.

N	1	2	3	4	5	6	7	8	9	10	11	12
S	0	1	1	3	1	6	1	7	4	8	1	16

(*Note that for* $N = 6$ *the sum also equals* 6. *For this reason,* 6 *is known as a 'perfect number'. After* 6, *the next two perfect numbers are* 28 *and* 496.)

11. **C** Opposite angles of a parallelogram are equal so $\angle QPS = 50°$. Therefore $\angle QPT = 112°$ and, as triangle *QPT* is isosceles, $\angle PQT = (180° - 112°) \div 2 = 34°$. As *PQRS* is a parallelogram, $\angle PQR = 180° - 50° = 130°$. So $\angle TQR = 130° - 34° = 96°$.

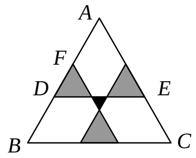
12. **B** The values of the expressions are £5.40, £6.00, £5.40, £5.40, £5.40 respectively.

13. **D** In the rally, approximately 90 shots were hit per minute for a total of 132 minutes. As $90 \times 130 = 11\,700$, D is the best alternative.

14. **A** The mean of the first three numbers is $\frac{1}{3}(20 + x)$; the mean of the last four numbers is $\frac{1}{4}(33 + x)$. Therefore $4(20 + x) = 3(33 + x)$, that is $80 + 4x = 99 + 3x$, so $x = 99 - 80 = 19$.

15. **E** $\frac{1}{2} + \frac{1}{3} \times \frac{1}{4} = \frac{1}{2} + \frac{1}{12} = \frac{7}{12}$; $\frac{1}{2} + \frac{1}{3} \div \frac{1}{4} = \frac{1}{2} + \frac{1}{3} \times \frac{4}{1} = \frac{1}{2} + \frac{4}{3} = \frac{11}{6}$; $\frac{1}{2} \times \frac{1}{3} \div \frac{1}{4} = \frac{1}{2} \times \frac{1}{3} \times \frac{4}{1} = \frac{2}{3}$; $\frac{1}{2} - \frac{1}{3} \div \frac{1}{4} = \frac{1}{2} - \frac{1}{3} \times \frac{4}{1} = \frac{1}{2} - \frac{4}{3} = -\frac{5}{6}$; $\frac{1}{2} - \frac{1}{3} \times \frac{1}{4} = \frac{1}{2} - \frac{1}{12} = \frac{5}{12}$.
Of the fractions $\frac{7}{12}, \frac{11}{6}, \frac{2}{3}, -\frac{5}{6}, \frac{5}{12}$, the closest to 0 is $\frac{5}{12}$.

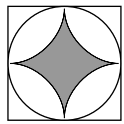
16. **B** As triangle ABC is equilateral, $\angle BAC = 60°$. From the symmetry of the figure, we may deduce that $AD = DE$ so triangle ADE is equilateral. The length of the side of this equilateral triangle = length of DE = (5 + 2 + 5) cm =12 cm. So $AF = AD - AF = (12 - 5)$ cm $= 7$ cm. By a similar argument, we deduce that BD = 7 cm, so the length of the side of triangle $ABC = (7 + 5 + 7)$ cm $= 19$ cm.



17. **E** The terms of the sequence are 6, 3, 14, 7, 34, 17, 84, 42, 21, 104, 52, 26, 13, 64, 32, 16, 8, 4, 2, 1, 4, 2, 1, … . As can be seen, there will now be no other terms in the sequence other than 4, 2 and 1. It can also be seen that the only values of n for which the n th term = n are 13 and 16.

18. **C** After traversing the first semicircle, Peri will be at the point (8, 0); after the second semicircle Peri will be at (4, 0) and after the third semicircle, Peri will be at the point (2, 0).

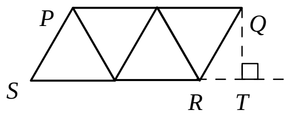
19. **A** The diagram shows the original diagram enclosed within a square of side $2r$, where r is the radius of the original circle. The unshaded area of the square consists of four quadrants (quarter circles) of radius r. So the shaded area is $4r^2 - \pi r^2 = r^2(4 - \pi)$. Therefore the required fraction is



$$\frac{r^2(4 - \pi)}{\pi r^2} = \frac{4 - \pi}{\pi} = \frac{4}{\pi} - 1.$$

20. **D** Let the sides of the rectangle, in cm, be $4x$ and $5x$ respectively.
Then the area of the square is $4x \times 5x\,\text{cm}^2 = 20x^2\,\text{cm}^2$. So $20x^2 = 125$, that is $x^2 = \frac{25}{4}$. Therefore $x = \pm\frac{5}{2}$, but x cannot be negative so the sides of the rectangle are 10 cm and 12.5 cm. Hence the rectangle has perimeter 45 cm.

21. **A** In the diagram, T is the foot of the perpendicular from Q to SR produced. Angles PQR and QRT are alternate angles between parallel lines so $\angle QRT = 60°$. Triangle QRT has interior angles of 90°, 60°, 30° so it may be thought of as being half of an equilateral triangle of side 1 unit, since the length of QR is 1 unit. So the lengths of RT and QT are $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$ units respectively.



Applying Pythagoras' Theorem to ΔQST, $SQ^2 = ST^2 + QT^2 = \left(\frac{5}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{25}{4} + \frac{3}{4} = 7$.
So the length of SQ is $\sqrt{7}$ units.

22. E The median number of cups of coffee is the median of a sequence of 190 positive integers $(t_1, t_2, \ldots, t_{190})$. Let the sum of these terms be S.

The median of the 190 numbers is $\frac{1}{2}(t_{95} + t_{96})$. The alternatives imply that the median cannot be greater than 3.5. The next lowest possible value for the median would be 4. For this to be possible, $t_{95} + t_{96} = 8$.

If $t_{95} = t_{96} = 4$ then the minimum value for S would occur if all other values of t were as small as possible, that is the first 94 values would all equal 1 and the last 94 values would all equal 4. In this case, $S = 94 \times 1 + 2 \times 4 + 94 \times 4 = 478$, whereas we are told that 477 cups of coffee were sold. Any other values of t_{95} and t_{96} such that $t_{95} + t_{96} = 8$ would produce a larger minimum value of S. For example, if $t_{95} = 3$ and $t_{96} = 5$ then the minimum value of S would be $94 \times 1 + 3 + 5 + 94 \times 5$, that is 572. So the median of the 190 terms cannot be 4, but it is possible for it to be 3.5. If the first 94 terms all equal 1, $t_{95} = 3$ and $t_{96} = 4$ and the last 94 terms all equal 4 then $S = 477$ as required and the median is $\frac{1}{2}(3 + 4) = 3.5$.

So the maximum possible value of the median number of cups of coffee bought per customer is 3.5.

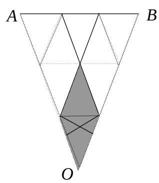
23. C As in the solution for Q21, ΔPTS may be thought of as half an equilateral triangle, so TS has length 1 unit. Therefore ΔSRT is isosceles and, as $\angle TSR = 120°$, $\angle SRT = \angle STR = 30°$. So $\angle TRQ = 45° - 30° = 15°$. Using the exterior angle theorem in ΔTQR, $\angle TQR = \angle STR - \angle TRQ = 30° - 15° = 15°$. So ΔTQR is isosceles with $TQ = TR$. However, ΔPRT is also isosceles with $PT = TR$ since $\angle PRT = \angle TPR = 30°$. Therefore $TQ = TP$, from which we deduce that PQT is an isosceles right-angled triangle in which $\angle PQT = \angle QPT = 45°$. So $\angle QPR = \angle QPT + \angle TPS = 45° + 30° = 75°$.

24. D The nature of the spiral means that 4 is in the top left-hand corner of a 2×2 square of cells, 9 is in the bottom right-hand corner of a 3×3 square of cells, 16 is in the top left-hand corner of a 4×4 square of cells and so on. To find the position of 2012 in the grid, we note that $45^2 = 2025$ so 2025 is in the bottom right-hand corner of a 45×45 square of cells and note also that $47^2 = 2209$. The table below shows the part of the grid in which 2012 lies. The top row shows the last 15 cells in the bottom row of a 45×45 square of cells, whilst below it are the last 16 cells in the bottom row of a 47×47 square of cells.

2011	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022	2023	2024	2025	
2194	2195	2196	2197	2198	2199	2200	2201	2202	2203	2204	2205	2206	2207	2208	2209

So 2195 lies below 2012.

25. B The diagram shows part of the ceramic.



A and B are vertices of the outer octagon, which has O at its centre. The solid lines are part of the original figure, whilst the broken lines OA, OB, two broken lines which are parallel to AB and broken lines parallel to OA and OB respectively have been added. As can be seen, these lines divide ΔOAB into nine congruent triangles. The shaded portion of triangle has area equal to that of two of the triangles. So $\frac{2}{9}$ of the area of ΔOAB has been shaded. Now the area of the outer octagon is eight times the area of ΔOAB and the area of shaded portion of the design is eight times the area of the shaded portion of ΔOAB so the fraction of the octagon which is shaded is also $\frac{2}{9}$.