UK INTERMEDIATE MATHEMATICAL CHALLENGE

February 3rd 2011

SOLUTIONS

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some ***Extension Problems*** for further investigations.

The Intermediate Mathematical Challenge (IMC) is a multiple choice contest, in which you are presented with five alternative answers, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the IMC, and we often give first a solution using this approach.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. So usually we have included a complete solution which does not use the fact that one of the given alternatives is correct. (A few questions do not lend themselves to such a treatment.) Thus we have aimed to give full solutions with all steps explained. We therefore hope that these solutions can be used as a model for the type of written solution that is expected in the Intermediate Mathematical Olympiad and similar competitions.

We welcome comments on these solutions, and, especially, corrections or suggestions for improving them. Please send your comments,

either by e-mail to

enquiry@ukmt.co.uk

or by post to

IMC Solutions, UKMT Maths Challenges Office, School of Mathematics,
University of Leeds, Leeds LS2 9JT.

Quick Marking Guide

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
B	A	E	E	C	B	A	E	A	E	E	D	D	E	A	C	C	D	B	D	B	B	A	C	D

1.	What is the value of $4.5 \times 5.5 + 4.5 \times 4.5$?				
	A 36.5	B 45	C 50	D 90	E 100

Solution: **B**

Since you are not allowed to use a calculator in the IMC, it is a good idea to look for a way to avoid having to do multiplication sums. The presence of the factor 4.5 in both products provides a clue to an efficient method. If we take out this common factor, we obtain

$$4.5 \times 5.5 + 4.5 \times 4.5 = 4.5 \times (5.5 + 4.5) = 4.5 \times 10 = 45 .$$

Extension Problem.

1. *Without using a calculator* find the value of $123.4 \times 123.4 + 123.4 \times 876.6$

2.	To find the diameter in mm of a Japanese knitting needle, you multiply the size by 0.3 and add 2.1. What is the diameter in mm of a size 5 Japanese knitting needle?				
	A 3.6	B 7.4	C 10.8	D 12	E 17.1

Solution: **A**

We need to work out $5 \times 0.3 + 2.1$. This gives $1.5 + 2.1 = 3.6$ as the answer.

3.	The consecutive digits $1, 2, 3, 4$ in that order can be arranged to make the correct division, $12 \div 3 = 4$. One *other* sequence of four consecutive digits p, q, r, s makes a correct division, $'pq' \div r = s$. What is the value of s in this case?				
	A 4	B 5	C 6	D 7	E 8

Solution: **E**

In the context of the IMC it is good enough to try out all the cases in turn until we find a correct sum. We see that $23 \div 4 \neq 5$, $34 \div 5 \neq 6$, $45 \div 6 \neq 7$ but $56 \div 7 = 8$.

To check that this really gives the only sequence that gives a correct division sum you could just check the remaining case $p = 6$. However, it is more illuminating to tackle the question algebraically.

Suppose the digits p, q, r and s are consecutive, then $q = p + 1$, $r = p + 2$ and $s = p + 3$. Now 'pq' represents the number $10p + q$, that is, the number $10p + (p + 1) = 11p + 1$. So the equation 'pq' $\div r = s$ is equivalent to $(11p + 1) \div (p + 2) = p + 3$.

We now have that, as $p + 2 \neq 0$,

$(11p + 1) \div (p + 2) = p + 3$ if and only if $11p + 1 = (p + 2)(p + 3)$,

if and only if $11p + 1 = p^2 + 5p + 6$,

if and only if $p^2 - 6p + 5 = 0$,

if and only if $(p - 1)(p - 5) = 0$.

This gives just the two solutions $p = 1,5$ corresponding to the correct equations $12 \div 3 = 4$ and $56 \div 7 = 8$.

4. The angles of a triangle are in the ratio 2:3:5. What is the difference between the largest angle and the smallest angle?

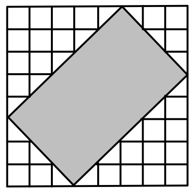
A 9° B 18° C 36° D 45° E 54°

Solution: **E**

The sum of the angles in a triangle is 180°. Since the angles are in the ratio 2:3:5 we have to divide 180° in the ratio 2:3:5. The difference between the largest angle the smallest angle is therefore $\frac{5-2}{2+3+5} \times 180^\circ = \frac{3}{10} \times 180^\circ = 54^\circ$.

5. The diagram shows a rectangle placed on a grid of 1 cm × 1 cm squares. What is the area of the rectangle in cm^2?

A 15 B $22\frac{1}{2}$ C 30 D 36 E 45



Solution: **C**

The total area of the 8×8 grid in cm^2 is $8^2 = 64$. The two larger triangles in the top left and bottom right corners of the grid make up a 5×5 square with area 25 cm^2. The two smaller triangles in the other corners of the grid make up a 3×3 square with area 9 cm^2. Hence the area of the rectangle, in cm^2, is $64 - 25 - 9 = 30$.

6. When I glanced at my car milometer it showed 24942, a palindromic number. Two days later, I noticed that it showed the next palindromic number. How many miles did my car travel in those two days?

A 100 B 110 C 200 D 220 E 1010

Solution: **B**

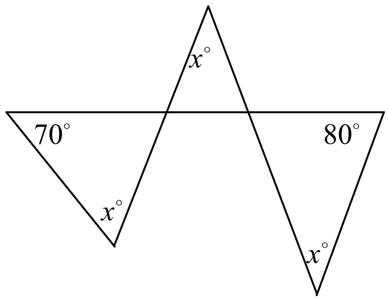
The only 5-digit palindromic number that begins 249 is 24942, so the next palindromic must be greater than 24999. The only 5-digit palindromic number beginning 250 is 25052, so this is the next palindromic number after 24942. We have that $25052 - 24942 = 110$, and so the answer is 110.

Extension problems:

1. Which is the next palindromic number after 25052?
2. How many 5-digit palindromic numbers are there?
3. Can you find a formula for the number of *n*-digit palindromic numbers?
   [Hint: you may need to consider the cases where *n* is even, and where *n* is odd, separately.]

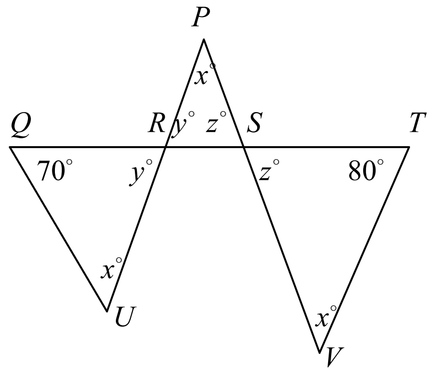
7. What is the value of x in this diagram?

A 30 B 35 C 40 D 45 E 50



Solution: **A**

We label the points P, Q, R, S, T, U and V as shown.



The vertically opposite angles $\angle PRS$ and $\angle QRU$ are equal, as are $\angle PSR$ and $\angle TSV$. We let these angles be y° and z°, as shown in the diagram.
We can find the value of x in more than one way.

1. Since $\angle QUP = \angle UPV$, the lines QU and PV are parallel. Hence the corresponding angles $\angle RQU$ and $\angle TSV$ are equal. So $z = 70$. Since the angles of a triangle add up to 180^0, we have from triangle STV that $x = 180 - 70 - 80 = 30$.

2. From the triangles PRS, QRU and STV we have that

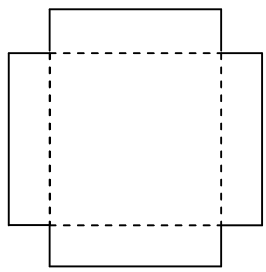
$$x + y + z = 180, \qquad (1)$$
$$x + y + 70 = 180, \qquad (2)$$
and
$$x + z + 80 = 180. \qquad (3)$$

From (1) and (2), $z = 70$ and hence, from (3), $x = 30$.

8. A square piece of card has a square of side 2 cm cut out from each of its corners. The remaining card is then folded along the dotted lines shown to form an open box whose total internal surface area is 180 cm^2.

What is the volume of the open box in cm^3?

A 100 B 128 C 162 D 180 E 200



Solution: **E**

Suppose that the dotted square corresponding to the base of the box has side length x cm. Then the base has area x^2 cm, and each of the sides of the box has area $2x$ cm. Hence the internal surface area of the box, in cm^2, is $x^2 + 4 \times 2x$, that is, $x^2 + 8x$. Therefore $x^2 + 8x = 180$. Now

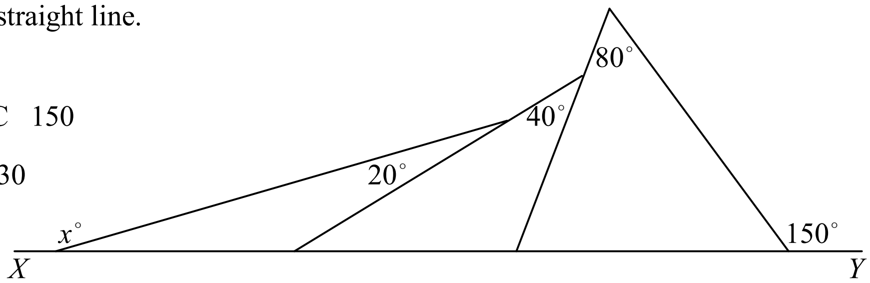
$$x^2 + 8x = 180 \text{ if and only if } x^2 + 8x - 180 = 0$$
$$\text{if and only if } (x+18)(x-10) = 0$$

with solutions $x = -18$ and $x = 10$. Since x must be positive, we deduce that $x = 10$.
The volume of the open box is the height multiplied by the area of the base, and, in cm^3, this is $2x^2$, that is, 200.

9. In the diagram, XY is a straight line.
What is the value of x?

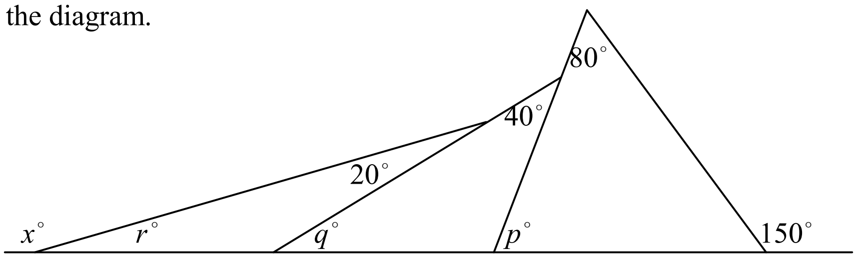
A 170 B 160 C 150

D 140 E 130



Solution: **A**

Let the angles be as marked in the diagram.



The exterior angle of a triangle is the sum of the two opposite interior angles. Therefore we have that

$$p + 80 = 150 \qquad (1)$$
$$q + 40 = p \qquad (2)$$
and
$$r + 20 = q \qquad (3)$$

From (1), $p = 70$. Hence, from (2), $q = 30$. Therefore, from (3), $r = 10$. Since the angles on a line add up to 180°, it follows that $x = 170$.

10. Merlin magically transforms a 6 tonne monster into mice with the same total mass.
Each mouse has a mass of 20g. How many mice does Merlin make?

A 30 B 300 C 3000 D 30 000 E 300 000

Solution: **E**

1 tonne = 1000 kg = 1000×1000 g, so 6 tonne = $6 \times 1000 \times 1000$ g. So the number of 20 g mice that Merlin makes is

$$\frac{6 \times 1000 \times 1000}{20} = 6 \times 1000 \times 50 = 6 \times 50\,000 = 300\,000.$$

11. What is the value of $19\frac{1}{2} \times 20\frac{1}{2}$?

A 250 B $380\frac{1}{4}$ C $390\frac{1}{4}$ D 395 E $399\frac{3}{4}$

Solution: **E**

Since in the IMC we are not allowed to use a calculator, we should look for a better method than just multiplying the two given numbers. The clue is that both numbers differ from 20 by $\frac{1}{2}$, so we can make use of the standard "difference of two squares" formula, $(a-b)(a+b) = a^2 - b^2$.

This gives $19\frac{1}{2} \times 20\frac{1}{2} = (20 - \frac{1}{2})(20 + \frac{1}{2}) = 20^2 - (\frac{1}{2})^2 = 400 - \frac{1}{4} = 399\frac{3}{4}$.

Extension problem.

1. Find the values of (a) $199\frac{1}{2} \times 200\frac{1}{2}$ and (b) $986\frac{2}{3} \times 1013\frac{1}{3}$ *without using a calculator*.

12. What is the sum of the first 2011 digits when $20 \div 11$ is written as a decimal?

A 6013 B 7024 C 8035 D 9046 E 10057

Solution: **D**

We see from the following long division sum

$$\begin{array}{r} 1.8181... \\ 11\overline{)20.0000...} \\ \underline{11} \\ 90 \\ \underline{88} \\ 20 \end{array}$$

that $20 \div 11$ has the recurring decimal expansion 1.8181…. . Hence the first 2011 digits consist of 1006 1s and 1005 8s. So the sum of these digits is $1006 \times 1 + 1005 \times 8 = 1006 + 8040 = 9046$.

13. The three blind mice stole a piece of cheese. In the night, the first mouse ate $\frac{1}{3}$ of the cheese. Later, the second mouse ate $\frac{1}{3}$ of the remaining cheese. Finally, the third mouse ate $\frac{1}{3}$ of what was then left of the cheese.
Between them, what fraction of the cheese did they eat?

A $\frac{16}{27}$ B $\frac{17}{27}$ C $\frac{2}{3}$ D $\frac{19}{27}$ E $\frac{20}{27}$

Solution: **D**

Our first method is a direct calculation: The first mice ate $\frac{1}{3}$ of the cheese, leaving $1 - \frac{1}{3} = \frac{2}{3}$. The second mouse ate $\frac{1}{3}$ of this, namely $\frac{1}{3} \times \frac{2}{3} = \frac{2}{9}$. This left $\frac{2}{3} - \frac{2}{9} = \frac{4}{9}$ of the cheese. The third mouse ate $\frac{1}{3}$ of this, namely $\frac{1}{3} \times \frac{4}{9} = \frac{4}{27}$. So between them, the amount of the cheese that the three mice ate was $\frac{1}{3} + \frac{2}{9} + \frac{4}{27} = \frac{9+6+4}{27} = \frac{19}{27}$.

Our second method is to first calculate the amount of cheese that remains. Each mouse eats $\frac{1}{3}$ rd of the remaining cheese, and so leaves $\frac{2}{3}$ rd of it. Hence, after the three mice have eaten the cheese, there remains $\left(\frac{2}{3}\right)^3$ of the initial amount of cheese. So the fraction which is eaten is $1 - \left(\frac{2}{3}\right)^3 = 1 - \frac{8}{27} = \frac{19}{27}$.

Extension problem.

1. Suppose there are n mice, and each, in turn, eats $\frac{1}{n}$ th of the remaining amount of cheese. Find a formula for the fraction of cheese that they eat between them.

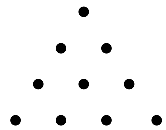
14. The number 6 lies exactly halfway between 3 and 3^2. Which of the following is not halfway between a positive integer and its square?

A 3 B 10 C 15 D 21 E 30

Solution: **E**

The number k lies exactly halfway between the positive integer n and its square n^2 if and only if $k = \dfrac{n + n^2}{2} = \dfrac{n(n+1)}{2}$. So the numbers of this form are, putting n successively equal to 1, 2, 3, … in this formula, the numbers in the sequence 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, … We see that of the alternative answers given, the only one that does not occur in this sequence is 30.

Note: You will probably recognize the numbers in the sequence 1, 3, 6, … as the *triangle* (or *triangular*) *numbers.* They are so-called because they are the number of dots in a triangular array such as



The number of dots in this array is $1+2+3+4$. In general a triangular number is a number which, for some positive integer *n*, is the sum of the first *n* positive integers. So, if we let T_n be the *n*th triangular number, $T_n = 1+2+3+...+n.$

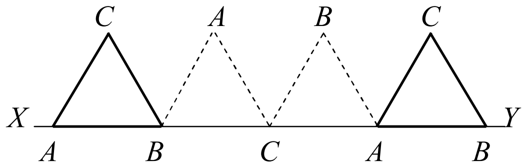
Extension Problems.

1. The formula for the sum $1+2+3+...+n$ is given by, $T_n = \dfrac{n(n+1)}{2}$. Can you prove that for each positive integer *n*, $1+2+3+...+n = \dfrac{n(n+1)}{2}$?

2. Since the sequence, 1, 3, 6, … of triangular numbers is increasing it is, in principle, possible to check whether a number is triangular by seeing whether or not it occurs in the sequence. However this is not very easy for a large number such as 6 126 750. Here is an alternative method. The positive integer *k* is triangular if and only $k = \dfrac{n^2+n}{2}$ for some positive integer *n*, that is, if and only if the quadratic equation $n^2+n-2k=0$ has a positive integer solution for *n.* Use the formula for the solutions of a quadratic equation to find a necessary and sufficient condition for *k* to be a triangular number. Then use this criterion to determine whether the following numbers are triangular
(a) 6 975 163, (b) 76 205 685.

3. Note that the sequence of triangular numbers, 1, 3, 6, 10, 15, 21, 28, 36, 45 ,… , includes two squares, 1 and 36. Find some more triangular numbers that are also squares. Are there infinitely many triangular numbers that are squares?

15. The equilateral triangle *ABC* has sides of length 1 and *AB* lies on the line *XY*. The triangle is rotated clockwise around *B* until *BC* lies on the line *XY*. It is then rotated similarly around *C* and then about *A* as shown in the diagram.

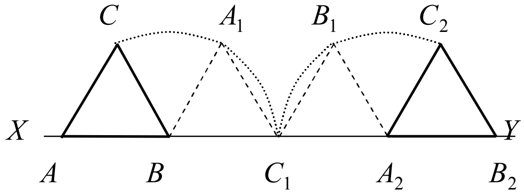


What is the length of the path traced out by point *C* during this sequence of rotations?

A $\dfrac{4\pi}{3}$ B $2\sqrt{3}$ C $\dfrac{8\pi}{3}$ D 3 E $\dfrac{2\pi}{3}$

Solution: **A**

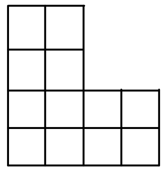
For convenience we have used A_1, A_2 etc for the subsequent positions taken up by *A*, *B* and *C*.
C first moves along a circular arc with centre *B*.
This arc has radius 1. Since $\angle CBC_1 = 120^0$, which corresponds to one-third of a complete turn, the length of this arc is $\frac{1}{3} \times 2\pi = \frac{2}{3}\pi$. When the triangle is rotated about the point C_1 the vertex *C* does not move at all. Finally, when the triangle is rotated about the point A_2 , *C* again turns through an angle 120^0. and so again moves along an arc of length $\frac{2}{3}\pi$. Therefore the total length of the path traced out by *C* is $\frac{2}{3}\pi + \frac{2}{3}\pi = \frac{4}{3}\pi$.



16. The diagram shows an L-shape divided into 1×1 squares. Gwyn cuts the shape along some of the lines shown to make two pieces neither of which is a square. She then uses the pieces to form a 2×6 rectangle.

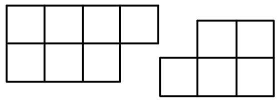
What is the difference between the areas of the two pieces?

A 0 B 1 C 2 D 3 E 4



Solution: **C**

The L-shape needs to be divided as shown if Gwyn is to make a 2×6 rectangle from two pieces which are not squares. Note that one piece must be turned over. The pieces have areas 7 and 5, whose difference is 2.



17. A shop advertised "Everything half price in our sale", but also now advertises that there is "An additional 15% off sale prices". Overall, this is equivalent to what reduction on the original prices?

A 7.5% B 35% C 57.5% D 65% E 80%

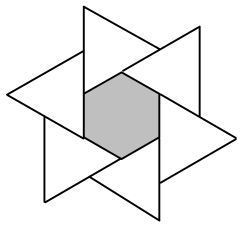
Solution: **C**

To reduce prices by a half, we multiply the price by 0.5. To reduce prices by a further 15%, we then multiply by 0.85. Therefore the final price is obtained by multiplying the original price by $0.5 \times 0.85 = 0.425$. So the final price is 42.5% of the original price. So the reduction is 57.5%.

18. The diagram contains six equilateral triangles with sides of length 2 and a regular hexagon with sides of length 1.

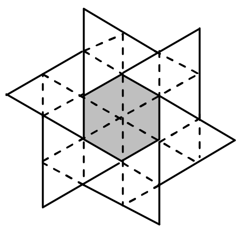
What fraction of the whole shape is shaded?

A $\frac{1}{8}$ B $\frac{1}{7}$ C $\frac{1}{6}$ D $\frac{1}{5}$ E $\frac{1}{4}$



Solution: **D**

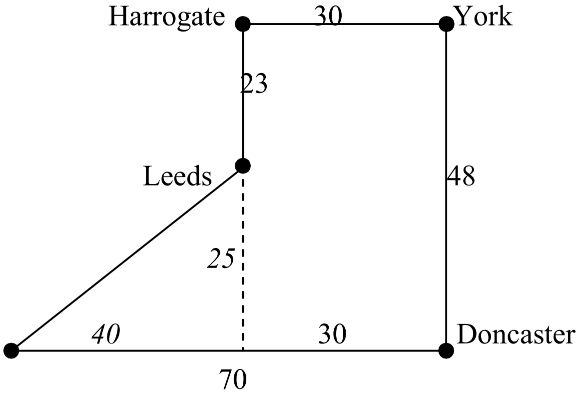
Each of the equilateral triangles with side length 2, may be divided into 4 equilateral triangles, each with side length 1, as shown.
In a similar way, the regular hexagon may be divided up into 6 equilateral triangles each with side length 1. So the whole shape is made up of $6 \times 4 + 6 = 30$ of these congruent small triangles, of which 6 are shaded. So the proportion of the figure that is shaded is $\frac{6}{30} = \frac{1}{5}$.



19. Harrogate is 23 km due north of Leeds, York is 30 km due east of Harrogate, Doncaster is 48 km due south of York, and Manchester is 70 km due west of Doncaster. To the nearest kilometre, how far is it from Leeds to Manchester, as the crow flies?

A 38km B 47km C 56km D 65km E 74km

Solution: **B**

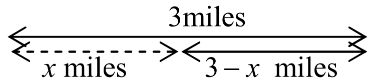


We see from the diagram that Manchester is 40 km west of Leeds, and 25 km south of Leeds. So, by Pythagoras' Theorem, the distance between them is, in km, $\sqrt{25^2+40^2}=5\sqrt{5^2+8^2}=5\sqrt{89}$.
Since $81<89<100$, we have that $9<\sqrt{89}<10$, and therefore $45<5\sqrt{89}<50$. So of the options given, 47 is the best approximation.
[To check that this really is the best integer approximation to $5\sqrt{89}$. you would need to check that, $9.4<\sqrt{89}<9.5$.
To do this you need to check, by a direct calculation, that $9.4^2<89<9.5^2$.]

20. Max and his dog Molly set out for a walk. Max walked up the road and then back down again, completing a six mile round trip. Molly, being an old dog, walked at half Max's speed. When Max reached the end of the road, he turned round and walked back to the starting point, at his original speed. Part way back he met Molly, who then turned around and followed Max home, still maintaining her original speed. How far did Molly walk?

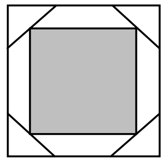
A 1 mile B 2 miles C 3 miles D 4 miles E 5 miles.

Solution: **D**



Max completes a 6 mile trip, so he walks 3 miles before turning round. Suppose that he meets Molly after Molly has walked x miles. So when they meet Max has covered $3-x$ miles of his return journey, and so he has walked $3+(3-x)=6-x$ miles while Molly has walked x miles. Since Molly walks at half Max's speed, $x=\frac{1}{2}(6-x)$. So $x=3-\frac{1}{2}x$. Hence $\frac{3}{2}x=3$, and so $x=2$. Molly altogether walks $x+x$ miles, that is, 4 miles.

21. A regular octagon is placed inside a square, as shown. The shaded square connects the midpoints of the four sides of the octagon.

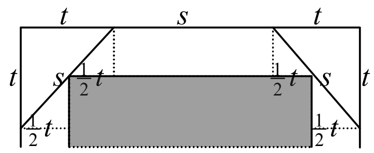


What fraction of the outer square is shaded?

A $\sqrt{2}-1$ B $\frac{1}{2}$ C $\frac{\sqrt{2}+1}{4}$ D $\frac{\sqrt{2}+2}{5}$ E $\frac{3}{4}$

Solution: **B**

Suppose that the side length of the regular octagon is s units. The right angled isosceles triangles in the diagram have hypotenuses of length s. Suppose that the length of each of the other sides of these triangles is t. By Pythagoras' Theorem, $t^2+t^2=s^2$, and hence $t=\frac{1}{\sqrt{2}}s$. Hence the side length of the larger square is $s+2t=s+\sqrt{2}s=(1+\sqrt{2})s$.



The side length of the smaller square is $s+\frac{1}{2}t+\frac{1}{2}t=s+t$ $=s+\frac{1}{\sqrt{2}}s=\frac{1}{\sqrt{2}}(\sqrt{2}+1)s=\frac{1}{\sqrt{2}}(1+\sqrt{2})s$.

Hence the fraction of the outer square that is shaded is

$$\frac{\left(\frac{1}{\sqrt{2}}(1+\sqrt{2})s\right)^2}{((1+\sqrt{2}s)^2}=\left(\frac{1}{\sqrt{2}}\right)^2=\frac{1}{2}.$$

22. You are given that $5^p=9$, $9^q=12$, $12^r=16$, $16^s=20$ and $20^t=25$. What is the value of $pqrst$?

A 1 B 2 C 3 D 4 E 5

Solution: **B**

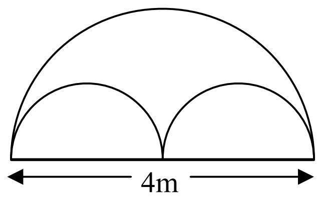
The key to the solution is to first calculate the value of 5^{pqrst}. We have that

$$5^{pqrst}=(5^p)^{qrst}=9^{qrst}=(9^q)^{rst}=12^{rst}=(12^r)^{st}=16^{st}=(16^s)^t=20^t=25=5^2.$$

Hence $pqrst=2$

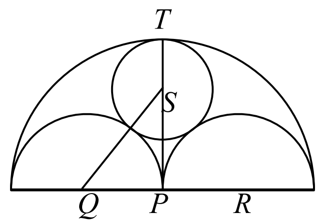
23. A window frame in Salt's Mill consists of two equal semicircles and a circle inside a large semicircle with each touching the other three as shown. The width of the frame is 4m.

What is the radius of the circle, in metres?



A $\frac{2}{3}$ B $\frac{\sqrt{2}}{2}$ C $\frac{3}{4}$ D $2\sqrt{2}-1$ E 1

Solution: **A**



Suppose that the radius of the circle is x m.
We let P be the centre of the large semicircle, and Q and R be centres of the smaller semicircles. We let S be the centre of the circle and T the point where PS meets the large semicircle. The large semicircle has radius 2 m and the smaller semicircles have radius 1 m.

So we have that PQ has length 1, PS has length $2-x$ and SQ has length $1+x$. PS is a tangent to the smaller semicircles, and therefore $\angle SPQ = 90^0$. So, applying Pythagoras' Theorem to the triangle PQS, we have

$$1^2 + (2-x)^2 = (1+x)^2$$

that is $$1 + 4 - 4x + x^2 = 1 + 2x + x^2$$

and hence $$4 = 6x$$

and therefore $$x = \frac{2}{3}.$$

Notes:

1. It follows that in the right angled triangle QPS the side lengths are in the ratio $1:2-\frac{2}{3}:1+\frac{2}{3}$, that is, 3:4:5.
2. Salt's Mill was textile mill, built by Titus Salt in 1853 at Saltaire near Bradford. It was bought by Jonathan Silver in 1987. Today it is centre for art and commerce, with the Hockney art galleries, restaurants, shopping and office space. You will find more information about it on its website: http://www.saltsmill.org.uk/ .

24. Given any positive integer n, Paul adds together the distinct factors of n, other than n itself. Which of these numbers can never be Paul's answer?

A 1 B 3 C 5 D 7 E 9

Solution: **C**

The only factor of 2, other than 2 itself is 1. So the sum of these factors is 1. [Note that whenever p is a prime number, the sum of the factors of p, other than p, is 1.]

The factors of 4, other than 4 itself, are 1 and 2, whose sum is 3.

The factors of 8, other than 8 itself, are 1, 2, 4, whose sum is 7.

The factors of 15, other than 15 itself, are 1, 3, 5, whose sum is 9.

So options A, B, D and E can each be Paul's answer.

In the context of the IMC this is enough for us to be able to select C as the correct option. However, to give a mathematically complete answer, we need to give a reason why the sum of the factors of n, other than n itself, cannot equal 5.

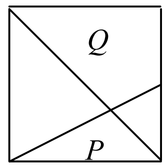
Clearly, we need only consider the case where $n > 1$. So, one of the factors of n, other than n itself, is 1. Suppose that the other factors are $a, b, \ldots$. Then $a, b, \ldots$ are distinct, none of them is 1, and $1 + a + b + \ldots = 5$. So $a + b + \ldots = 4$. However, there is no way of expressing 4 as the sum of more than one distinct positive integer none of which is 1. So the only possibility is that 4 is the only factor of n, other than 1 and n. However, this is impossible, since if 4 is a factor of n, then so also is 2. Therefore option C is not possible.

Extension Problems.

1. Investigate the sums of the factors of n, other than n itself, for all the positive integers in the range $1 \le n \le 32$ that are not prime numbers.
2. We have shown above that 5 does not occur as the sum of the factors of n other than n itself. Find another value that does not occur.
3. Prove that the sum of the factors of a positive integer n, other than n itself, is never 52.

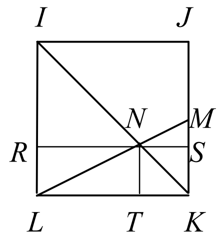
25. The diagram shows a square, a diagonal and a line joining a vertex to the midpoint of a side. What is the ratio of area P to area Q?

A $1:\sqrt{2}$ B $2:3$ C 1:2 D 2:5 E 1:3



Solution: **D**

Let I, J, K and L be the vertices of the square. Let M be the midpoint of JK, let N be the point where the diagonal IK meets LM. Let the line through N parallel to LK meet IL at R and JK at S. Let T be the foot of the perpendicular from N to LK. Let the square have side length s.



In the triangles INL and KNM, the opposite angles $\angle INL$ and $\angle KNM$ are equal. Also, as IL is parallel to JK, the alternate angles $\angle LIN$ and $\angle MKN$ are equal. Therefore the triangles INL and KNM are similar. Hence $\dfrac{IN}{NK} = \dfrac{IL}{MK} = 2$. Similarly, the triangles INR and KNS are similar. Therefore $\dfrac{NR}{NS} = \dfrac{IN}{NK} = 2$. So $NR = \frac{2}{3}s$ and $NS = \frac{1}{3}s$.

Now, $NTKS$ is square, because its angles are all right-angles, and $\angle NKT = 45^\circ$. Therefore $NT = NS = \frac{1}{3}s$.

It follows that the area of the triangles LNK, INL and MNK are $\frac{1}{2}(s \times \frac{1}{3}s) = \frac{1}{6}s^2$, $\frac{1}{2}(s \times \frac{2}{3}s) = \frac{1}{3}s^2$, and $\frac{1}{2}(\frac{1}{2}s \times \frac{1}{3}s) = \frac{1}{12}s^2$, respectively.

The area of the region P, is that of the triangle LNK, that is, $\frac{1}{6}s^2$. The area of the region Q is obtained by subtracting the areas of the triangles LNK, INL and MNK from the area of the square. So region Q has area $s^2 - \frac{1}{6}s^2 - \frac{1}{3}s^2 - \frac{1}{12}s^2 = \frac{5}{12}s^2$. So the ratio of these areas is $\frac{1}{6}s^2 : \frac{5}{12}s^2 = \frac{1}{6} : \frac{5}{12} = 2:5$.