United Kingdom
Mathematics Trust

Intermediate Mathematical Challenge

Solutions 2021

For reasons of space, these solutions are necessarily brief.

There are more in-depth, extended solutions available on the UKMT website, which include some exercises for further investigation:

www.ukmt.org.uk

1. B $2021 - 2223 + 2425 = 2021 + 2425 - 2223 = 4446 - 2223 = 2223$.

2. A Today is Thursday, so the day before the day before yesterday was Monday. This was two days after the day before my birthday, hence the day before my birthday was Saturday. Therefore my birthday was on Sunday.

3. C $2 - (-2 - 2) - (-2 - (-2 - 2)) = 2 - (-4) - (-2 - (-4)) = 2 + 4 - (-2 + 4) = 2 + 4 - 2 = 4$.

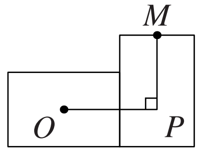
4. A Note that $PQYXWVU$ is a heptagon (seven-sided polygon).
Therefore the sum of its interior angles $= (7 - 2) \times 180° = 900°$.
Since $PQRS$ is a square, $\angle UPQ = \angle PQY = 90°$.
Also, since $TUVW$ and $WXYZ$ are squares, reflex $\angle UVW =$ reflex $\angle WXY = 270°$.
So $\angle VWX = (900 - 2 \times 90 - 2 \times 270 - 62 - 74)° = 44°$.

5. D Let June have x sweets. Then May has $\frac{3x}{4}$ sweets. Also, April has $\frac{2}{3} \times \frac{3x}{4}$, that is $\frac{x}{2}$, sweets.
Therefore $x + \frac{3x}{4} + \frac{x}{2} = 90$. So $\frac{9x}{4} = 90$. Hence $x = \frac{4}{9} \times 90 = 40$.

6. E In ascending order, the first factors of 240 are 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 20.
So the first numbers on Kai's list are 7, 9, 11, 13, 14, 17, 18, 19.
Hence the sixth number on the list is 17.

7. C $(4 - \frac{1}{4}) \div (2 - \frac{1}{2}) = \frac{15}{4} \div \frac{3}{2} = \frac{15}{4} \times \frac{2}{3} = \frac{5}{2} = 2\tfrac{1}{2}$.

8. B Let P be a point inside one of the rectangles such that OP is parallel to the horizontal edges of the rectangles and MP is parallel to the vertical edges.
Therefore $\angle OPM$ is a right angle.



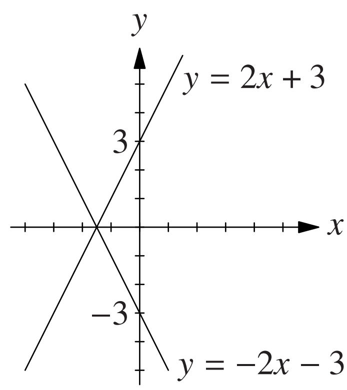
Then $OP = \frac{1}{2} \times 14 + \frac{1}{2} \times 10 = 7 + 5 = 12$.
Also, $MP = 14 - \frac{1}{2} \times 10 = 14 - 5 = 9$.

By Pythagoras' Theorem, $OM^2 = OP^2 + MP^2 = 12^2 + 9^2 = 144 + 81 = 225$.
Therefore $OM = \sqrt{225} = 15$.

9. **D** The first statement is false. For example, 2 and 3 are primes, but their product is 6 which is not prime.
Two squares may be denoted by m^2 and n^2, where m and n are non-negative integers.
Their product is $m^2n^2 = (mn)^2$. So a square multiplied by a square is always a square.
Two odd numbers may be denoted by $2m + 1$ and $2n + 1$, where m and n are integers.
Their product is $(2m+1)(2n+1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1$, which is also odd.
So an odd number multiplied by an odd number is always an odd number.
Two even numbers may be denoted by $2m$ and $2n$, where m and n are integers.
Their product is $4mn = 2(2mn)$, which is also even.
So an even number multiplied by an even number is always an even number.
Therefore three of the statements are true.

10. **D** Note that $53 \times 57 = (43 + 10)(47 + 10) = 43 \times 47 + 43 \times 10 + 10 \times 47 + 10 \times 10$
$= 2021 + (43 + 47 + 10) \times 10 = 2021 + 100 \times 10 = 2021 + 1000 = 3021$.

11. **E** The diagram shows that the reflection of the line $y = 2x + 3$ in the x-axis has gradient -2 and an intercept on the y-axis of $(0, -3)$.



Therefore, the equation of the reflected line is $y = -2x - 3$.

(*Note that reflection in the x-axis changes y to* $-y$.
Hence the equation of the reflected line is $-y = 2x + 3$,
so $y = -2x - 3$.)

12. **E** We are told that 40% of 50% of x is equal to 20% of 30% of y.
Therefore $0.4 \times 0.5 \times x = 0.2 \times 0.3 \times y$. Hence $0.2 \times x = 0.2 \times 0.3 \times y$.

So $y = \dfrac{x}{0.3} = \dfrac{10x}{3}$.

13. **A** Note that the sum of the digits of 12 345 is 15. So 12 345 is a multiple of 3.
Similarly, 54 321 is a multiple of 3. Therefore the product of these two integers is a multiple of 9.
Hence the remainder when $12\,345 \times 54\,321$ is divided by 9 is 0.

14. **B** Let the squares on the border of the large square have side-length 1. Then the large square has side-length 5 and area 25. Note that 8 of the 16 squares on the border of the large square are shaded. The area of the central region of the large square is $25 - 16 = 9$. Three-quarters of this area is shaded.

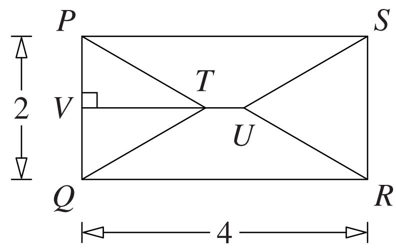
Therefore, the fraction of the large square which is shaded is $\dfrac{8 + \frac{3}{4} \times 9}{25} = \dfrac{32 + 27}{100} = \dfrac{59}{100}$.
So 59% of the large square is shaded.

15. **D** Let the distance, in miles, from P to Q be x.
Then the time taken to travel from P to Q is $\dfrac{x}{40}$ hours $= \dfrac{3x}{2}$ minutes.

Also, the time taken to travel from Q to P is $\dfrac{x}{45}$ hours $= \dfrac{4x}{3}$ minutes.
Therefore $\dfrac{3x}{2} - \dfrac{4x}{3} = 2$. So $\dfrac{x}{6} = 2$. Hence $x = 12$.

16. **E** The square has sides of length 2π, so its area is $(2\pi)^2 = 4\pi^2$. Each of the four semicircles has radius π, hence the sum of the areas of the semicircles is $4 \times \frac{1}{2} \times \pi \times \pi^2 = 2\pi^3$.
Therefore the required area is $2\pi^3 + 4\pi^2 = 2\pi^2(\pi + 2)$.

17. D In the diagram, V is the foot of the perpendicular from T to PQ.



Therefore, as PQT is an equilateral triangle, V is the midpoint of PQ.
By Pythagoras' Theorem in triangle VQT, $VT^2 + VQ^2 = QT^2$.
So $VT = \sqrt{2^2 - 1^2} = \sqrt{3}$.

Hence the area of triangle PQT is $\frac{1}{2} \times 2 \times \sqrt{3} = \sqrt{3}$. Similarly, the area of triangle $SRU = \sqrt{3}$.

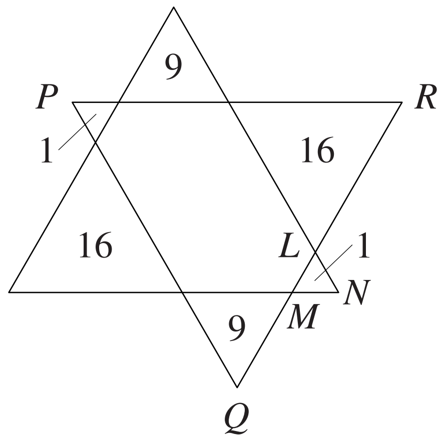
Therefore, as trapezia $QRUT$ and $PTUS$ are congruent, the area of trapezium $QRUT$ is $\frac{1}{2}\left(4 \times 2 - 2\sqrt{3}\right) = 4 - \sqrt{3}$.

18. C First note that $0.9\dot{5} < 0.\dot{9}\dot{5} < 0.\dot{9}6\dot{0} < 1 < 1.\dot{0}4\dot{0} < 1.\dot{0}\dot{5}$, so $0.\dot{9}6\dot{0}$ or $1.\dot{0}4\dot{0}$ is closest to 1.
Also, note that $0.\dot{9}6\dot{0} = 0.960960\ldots$, which differs from 1 by less than 0.04.
However, $1.\dot{0}4\dot{0} = 1.040040\ldots$, which differs from 1 by greater than 0.04.
So $0.\dot{9}6\dot{0}$ is closest in size to 1.

19. A The area of each rectangle is pq and the area of overlap is q^2, so the total area of the figure is $2pq - q^2$. Therefore $q^2 = \frac{1}{4}(2pq - q^2)$. Hence $4q^2 = 2pq - q^2$. So $5q^2 = 2pq$.
Now $q \neq 0$, so $5q = 2p$. Therefore $p : q = 5 : 2$.

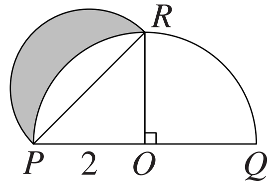
20. B Since the point (3, 1) lies on both lines, we have $1 = 3p + 4\ldots$ [1] and $p = 3q - 7 \ldots$ [2].
From [1], $3p = -3$, so $p = -1$. Substituting for p in [2] gives $-1 = 3q - 7$. Hence $3q = 6$.
So $q = 2$.

21. D The ratio of the areas of the three smaller equilateral triangles which have vertices at N, Q and R respectively is $1 : 9 : 16$. So the ratio of the side-lengths of these three triangles is $\sqrt{1} : \sqrt{9} : \sqrt{16} = 1 : 3 : 4$.
Hence the ratio of the side-length of triangle LMN to that of triangle PQR is $1 : 8$ and the ratio of the areas of these two triangles is $1^2 : 8^2 = 1 : 64$.
Therefore, as triangle LMN has area 1, triangle PQR has area 64.
So the area of the inner hexagon is $64 - (1 + 9 + 16) = 38$.



22. E $\left(1 + \frac{1}{x}\right)\left(1 - \frac{2}{x+1}\right)\left(1 + \frac{2}{x-1}\right) = \left(\frac{x+1}{x}\right)\left(\frac{x+1-2}{x+1}\right)\left(\frac{x-1+2}{x-1}\right) = \left(\frac{x+1}{x}\right)\left(\frac{x-1}{x+1}\right)\left(\frac{x+1}{x-1}\right)$
$= \left(\frac{x+1}{x}\right)$.

23. **B** By Pythagoras' Theorem, $PR^2 = PO^2 + RO^2 = 2^2 + 2^2 = 4 + 4 = 8$.
So $PR = \sqrt{8} = 2\sqrt{2}$.
Therefore the semicircle with diameter PR has radius $\sqrt{2}$.
Its area is $\frac{1}{2} \times \pi \times (\sqrt{2})^2 = \pi$.
The area of the quarter-circle bounded by PO, RO and arc PR is $\frac{1}{4} \times \pi \times 2^2 = \pi$.
The area of triangle POR is $\frac{1}{2} \times 2 \times 2 = 2$.
So the area of the segment bounded by PR and arc PR is $\pi - 2$.
Hence the area of the shaded region is $\pi - (\pi - 2) = 2$.



24. **B** The final sum of the numbers on the board is $1 + 2 + 3 + 4 + 5 + 6 + 5p + 7q = 21 + 5p + 7q$.
As there are now $6 + p + q$ numbers on the board, $\dfrac{21 + 5p + 7q}{6 + p + q} = 5.3$.
Therefore $10(21 + 5p + 7q) = 53(6 + p + q)$, that is $210 + 50p + 70q = 318 + 53p + 53q$.
So $17q = 108 + 3p = 3(36 + p)$. Hence q is a multiple of 3 as 17 and 3 are coprime.
Also, as $p > 0$, $17q > 108$. Therefore $q \geqslant 7$.
So the smallest possible value of q is 9; and this does satisfy the conditions with $p = 15$.

25. **E** Let the speed, in m/s, of the escalator when it is working be v. Let the length of the escalator be l m and let Thomas's running and walking speeds, in m/s, be r and w respectively.
As Thomas can run down the moving escalator in 15 seconds and walk down it in 30 seconds, $\dfrac{l}{v + r} = 15 \ldots [1]$; $\dfrac{l}{v + w} = 30 \ldots [2]$.
Also, as Thomas runs down the broken escalator in 20 seconds, $\dfrac{l}{r} = 20 \ldots [3]$.
From [1] $v + r = \dfrac{l}{15} \ldots [4]$; from [2] $v + w = \dfrac{l}{30} \ldots [5]$; from [3] $r = \dfrac{l}{20} \ldots [6]$.
Adding [5] and [6] and subtracting [4] gives $v + w + r - v - r = \dfrac{l}{30} + \dfrac{l}{20} - \dfrac{l}{15}$.
Therefore $w = \dfrac{l(2 + 3 - 4)}{60} = \dfrac{l}{60}$. So $\dfrac{l}{w} = 60$.

Hence it would take Thomas 60 seconds to walk down the broken escalator.