Intermediate Mathematical Challenge

Organised by the United Kingdom Mathematics Trust

supported by [XTX] MARKETS Overleaf

Solutions and investigations

February 2021

These solutions augment the shorter solutions also available online. The shorter solutions in many cases omit details. The solutions given here are full solutions, as explained below. In some cases alternative solutions are given. There are also many additional problems for further investigation. We welcome comments on these solutions and the additional problems. Please send them to enquiry@ukmt.org.uk.

The Intermediate Mathematical Challenge (IMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that sometimes you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the IMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. So for each question we have included a complete solution which does not use the fact that one of the given alternatives is correct. Thus we have aimed to give full solutions with each step explained (or, occasionally, left as an exercise). We therefore hope that these solutions can be used as a model for the type of written solution that is expected when a complete solution to a mathematical problem is required (for example, in the Intermediate Mathematical Olympiad and similar competitions).

Enquiries about the Intermediate Mathematical Challenge should be sent to:

IMC, UKMT, School of Mathematics Satellite, University of Leeds, Leeds LS2 9JT

☎ 0113 343 2339 enquiry@ukmt.org.uk www.ukmt.org.uk

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
B	A	C	A	D	E	C	B	D	D	E	E	A	B	D	E	D	C	A	B	D	E	B	B	E

1. What is the value of $2021 - 2223 + 2425$?

A 2122 B 2223 C 2324 D 2425 E 2526

SOLUTION **B**

COMMENTARY

Because $2021 - 2223 < 0$ this is a slightly awkward calculation to do without a calculator. We can get over this difficulty by changing the order of the numbers to avoid negative numbers. You might notice something that would make this calculation easier.

We have

$$\begin{aligned}2021 - 2223 + 2425 &= 2021 + 2425 - 2223\\ &= 4446 - 2223\\ &= 2 \times 2223 - 2223\\ &= 2223.\end{aligned}$$

FOR INVESTIGATION

1.1 What is the value of $222222 - 333333 + 444444$?

1.2 The value of $12345 - x + 54321$ is 33333. What is the value of x?

1.3 The value of $412347 - 444444 + x$ is 444444. What is the value of x?

2. The day before the day before yesterday was two days after the day before my birthday. Today is Thursday. On what day was my birthday?

A Sunday B Monday C Tuesday D Wednesday E Thursday

SOLUTION **A**

the day before my birthday	my birthday	the day before the day before yesterday	the day before yesterday	yesterday	today
Saturday	Sunday	Monday	Tuesday	Wednesday	Thursday

Today is Thursday. So yesterday was Wednesday. Thus the day before yesterday was Tuesday. Hence the day before the day before yesterday was Monday. Monday is two days after Saturday. So Saturday was the day before my birthday.

Hence my birthday was on Sunday.

3. What is the value of $2 - (-2 - 2) - (-2 - (-2 - 2))$?

A 0 B 2 C 4 D 6 E 8

Solution **C**

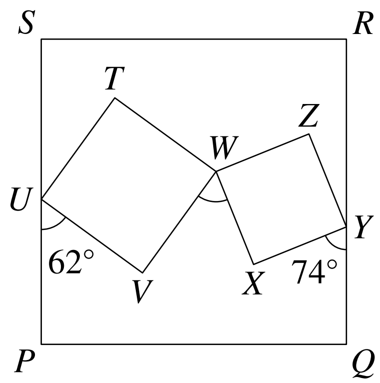
$$\begin{aligned}2 - (-2 - 2) - (-2 - (-2 - 2)) &= 2 - (-4) - (-2 - (-4))\\ &= 2 - (-4) - (-2 + 4)\\ &= 2 - (-4) - (2)\\ &= 2 + 4 - 2\\ &= 4.\end{aligned}$$

4. The diagram shows three squares, $PQRS$, $TUVW$ and $WXYZ$.

Angles PUV and QYX are 62° and 74° respectively.

What is angle VWX?

A 44° B 48° C 60° D 64° E 68°



Solution **A**

We note first that $PQYXWVU$ is a heptagon. In other words, it is a polygon with seven sides.

The sum of the internal angles of a polygon with n sides is $(n - 2) \times 180°$. [You are asked to prove this in Problem 4.1.] Hence the sum of the internal angles of $PQYXWVU$ is $(7 - 2) \times 180°$, that is $5 \times 180° = 900°$.

Because $PQRS$ is a square, the internal angles of the heptagon at P and Q are each 90°. Because $TUVW$ and $WXYZ$ are squares, the internal angles of the heptagon at X and V are each 270°.

Therefore, by adding the internal angles of the heptagon $PQYXWVU$, we have

$$90° + 90° + 74° + 270° + \angle VWX + 270° + 62° = 900°.$$

Hence

$$\begin{aligned}\angle VWX &= (900 - 90 - 90 - 74 - 270 - 270 - 62)°\\ &= (900 - 856)° = 44°.\end{aligned}$$

For investigation

4.1 Prove that the sum of the internal angles of a polygon with n sides is $(n - 2) \times 180°$.

5. April, May and June have 90 sweets between them. May has three-quarters of the number of sweets that June has. April has two-thirds of the number of sweets that May has. How many sweets does June have?

A 60 B 52 C 48 D 40 E 36

Solution **D**

Let J be the number of sweets that June has.

It follows that May has $\frac{3}{4}J$ sweets.

Hence April has $\frac{2}{3}(\frac{3}{4}J)$ sweets. So April has $(\frac{2}{3} \times \frac{3}{4})J = \frac{1}{2}J$ sweets.

Because April, May and June have 90 sweets between them,

$$90 = \tfrac{1}{2}J + \tfrac{3}{4}J + J = \left(\tfrac{1}{2} + \tfrac{3}{4} + 1\right)J = \tfrac{9}{4}J.$$

Thus

$$\tfrac{9}{4}J = 90$$

and we deduce that

$$J = \tfrac{4}{9} \times 90 = 40.$$

Hence June has 40 sweets.

6. Kai has begun to list, in ascending order, the positive integers which are *not* factors of 240.
What is the sixth number on Kai's list?

A 11 B 13 C 14 D 15 E 17

Solution **E**

The factors of 240 are, in ascending order,

$$1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, \ldots,$$

and therefore the positive integers that are not factors of 240 are, in ascending order,

$$7, 9, 11, 13, 14, 17, \ldots.$$

Therefore the sixth number on Kai's list is 17.

For investigation

6.1 Kai now begins to list, in ascending order, the positive integers which are *not* factors of 360. Which is the tenth number on Kai's new list?

6.2 Kai now lists all the positive integers that *are* factors of 600. What is the median of the numbers on this list?

6.3 Kai now lists all the positive integers that are factors of k^2, where is k a positive integer. What is the median of the numbers on this list?

7. What is the value of $(4 - \frac{1}{4}) \div (2 - \frac{1}{2})$?

A $1\frac{1}{2}$ B 2 C $2\frac{1}{2}$ D 3 E $4\frac{1}{4}$

SOLUTION **C**

We have

$$4 - \frac{1}{4} = \frac{16-1}{4} = \frac{15}{4}$$

and

$$2 - \frac{1}{2} = \frac{4-1}{2} = \frac{3}{2}.$$

Therefore

$$\begin{aligned}\left(4 - \frac{1}{4}\right) \div \left(2 - \frac{1}{2}\right) &= \frac{15}{4} \div \frac{3}{2} \\ &= \frac{15}{4} \times \frac{2}{3} \\ &= \frac{5}{2} \\ &= 2\tfrac{1}{2}.\end{aligned}$$

FOR INVESTIGATION

7.1 What is the value of

$$\left(9 - \frac{1}{9}\right) \div \left(3 - \frac{1}{3}\right).$$

7.2 Show that, in general, for $a \neq 0$,

$$\left(a^2 - \frac{1}{a^2}\right) \div \left(a - \frac{1}{a}\right) = a + \frac{1}{a}.$$

7.3 Find the value of

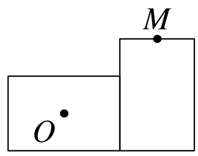
$$\left(64 - \frac{1}{64}\right) \div \left(4 - \frac{1}{4}\right).$$

7.4 Simplify the expression

$$\left(a^3 - \frac{1}{a^3}\right) \div \left(a - \frac{1}{a}\right).$$

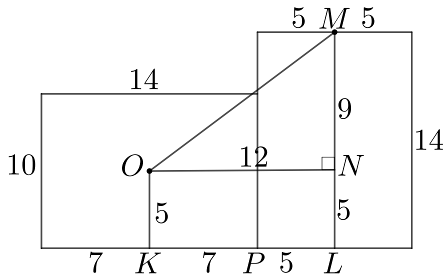
8. The diagram shows two 10 by 14 rectangles which are edge-to-edge and share a common vertex. It also shows the centre O of one rectangle and the midpoint M of one edge of the other.
What is the distance OM?

A 12 B 15 C 18 D 21 E 24



Solution **B**

Let K and L be the midpoints of the bases of the rectangles, as shown, let N be the point where the line through O parallel to KL meets ML and let P be the point on KL which is the common vertex of the two rectangles.



Then OK and ML are perpendicular to KL and ON is perpendicular to ML. It follows that $OKLN$ is a rectangle.

Hence we have $ON = KL = KP + PL = 7 + 5 = 12$.
Also, $MN = ML - NL = ML - OK = 14 - 5 = 9$.

Applying Pythagoras' Theorem to the right-angled triangle MNO, we have

$$OM^2 = MN^2 + NO^2 = 9^2 + 12^2 = 81 + 144 = 225 = 15^2.$$

It follows that the length of OM is 15.

Commentary

Note that the side lengths of the right angled triangle MNO are 9, 12 and 15. Thus MNO is just the basic 3-4-5 triangle scaled by the factor 3.

Three positive integers a, b and c which have the property that $a^2 + b^2 = c^2$ form what is called a *Pythagorean triple*. Thus 9, 12 and 15 form a Pythagorean triple. By Pythagoras' Theorem, Pythagorean triples are the side lengths of right-angled triangles with integer side lengths.

A Pythagorean triple a, b and c, which has no common factors other than 1 is called a *primitive Pythagorean triple*. For example the integers 3, 4 and 5 form a primitive Pythagorean triple, but 9, 12 and 15 do not.

For investigation

8.1 Find other examples of primitive Pythagorean triples.

8.2 Are there infinitely may different primitive Pythagorean triples? [To find the answer you may need to look in a book or use the internet.]

8.3 Does the line MO go through a vertex of one of the rectangles?

9. How many of the following statements are true?

A prime multiplied by a prime is always a prime.
A square multiplied by a square is always a square.
An odd number multiplied by an odd number is always an odd number.
An even number multiplied by an even number is always an even number.

A 0 B 1 C 2 D 3 E 4

Solution **D**

"A prime multiplied by a prime is always a prime." The product of the two primes 2 and 3 is 6 which is not a prime. This counterexample shows that this statement is false.

"A square multiplied by a square is always a square." We show that this statement is true. Let a and b be squares. Then there are positive integers m and n with $a = m^2$ and $b = n^2$. Then $a \times b = m^2 n^2 = (mn)^2$. Therefore $a \times b$ is also a square.

"An odd number multiplied by an odd number is always an odd number." We show that this statement is true. Let a and b be odd numbers. Then there are integers m and n with $a = 2m + 1$ and $b = 2n + 1$. Then $a \times b = (2m + 1)(2n + 1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1$. Since $2mn + m + n$ is an integer, $a \times b$ is also an odd number.

"An even number multiplied by an even number is always an even number." We show that this statement is true. Let a and b be even numbers. Then there are integers m and n with $a = 2m$ and $b = 2n$. Then $a \times b = 2m \times 2n = 2(2mn)$. Since $2mn$ is an integer, $a \times b$ is also an even number.

We therefore see that three of the statements given in the question are true.

For investigation

9.1 Which of the following statements are true?

(a) A composite number multiplied by a composite number is always a composite number.

(b) A cube multiplied by a cube is always a cube.

(c) A triangular number multiplied by a triangular number is always a triangular number.

(d) A number that is not a square multiplied by a number that is not a square is always a number that is not a square.

9.2 Which of the following statements are true?

(a) A prime number multiplied by a prime number is never a prime number.

(b) A triangular number multiplied by a triangular number is never a triangular number.

10. The prime factor decomposition of 2021 is 43×47.
What is the value of 53×57?

A 2221 B 2521 C 2921 D 3021 E 3031

SOLUTION **D**

COMMENTARY

If you have access to a calculator, this is a very easy question.

Even without a calculator, it is not difficult to use long multiplication to evaluate 53×57. The point of the question is that there is a better method making use of the given information that $43 \times 47 = 2021$.

Note also, that the factorization $2021 = 43 \times 47$ shows that the integer 2021 is the product of two consecutive prime numbers. Problem 10.3 asks you to find other year numbers with this property.

We have

$$\begin{aligned} 53 \times 57 &= (43 + 10) \times (47 + 10) \\ &= 43 \times 47 + 43 \times 10 + 10 \times 47 + 10 \times 10 \\ &= 2021 + 430 + 470 + 100 \\ &= 2021 + 1000 \\ &= 3021. \end{aligned}$$

FOR INVESTIGATION

10.1 The prime factorization of 869 is 11×79. What is the value of 21×89?

10.2 What is the value of each of the following?

(a) 143×147,

(b) 1043×1047,

(c) $10\,043 \times 10\,047$.

10.3 (a) Which is the last year number before 2021 which is the product of consecutive prime numbers?

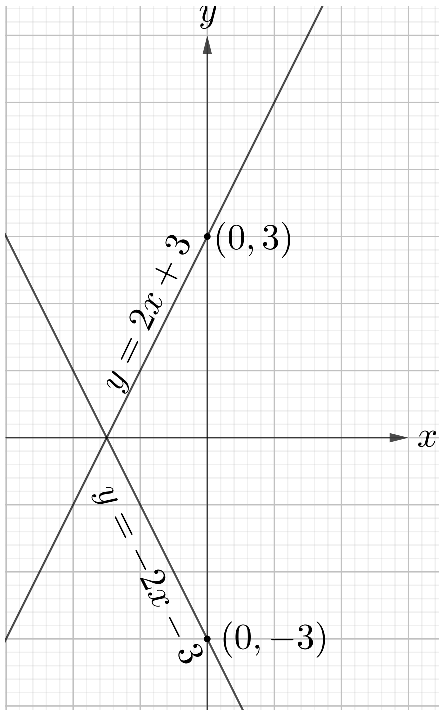
(b) Which is the next year number after 2021 which is the product of consecutive prime numbers?

(c) Which is the next year number after 2021 which is a prime?

11. The line with equation $y = 2x + 3$ is reflected in the x-axis.
Which of the following is the equation of the new line?

A $y = 2x - 3$ B $y = -2x + 3$ C $x = 2y + 3$ D $y = \frac{1}{2}x + 3$
E $y = -2x - 3$

Solution **E**



Method 1

The line with the equation $y = 2x + 3$ has gradient 2. It meets the y-axis at the point with coordinates $(0, 3)$.

Reflection in the x-axis maps this line to the line with gradient -2 which meets the y-axis at the point $(0, -3)$. The equation of this line is $y = -2x - 3$.

Method 2

Reflecting in the x-axis maps the point with coordinates (x, y) to the point with coordinates $(x, -y)$. So it maps the line with the equation $y = 2x + 3$ to the line with the equation $-y = 2x + 3$. This last equation is equivalent to the equation $y = -2x - 3$.

For investigation

11.1 The line with equation $y = 2x + 3$ is reflected in the y-axis. What is the equation of the new line?

11.2 The line with equation $y = 2x + 3$ is reflected in the line with the equation $y = x$. What is the equation of the new line?

11.3 The line with equation $y = 2x + 3$ is reflected in the line with equation $y = 2x$. What is the equation of the new line?

12. Andrew calculates that 40% of 50% of x is equal to 20% of 30% of y, where $x \neq 0$. Which of the following is true?

A $y = \frac{2x}{3}$ B $y = \frac{4x}{3}$ C $y = 2x$ D $y = \frac{8x}{3}$ E $y = \frac{10x}{3}$

SOLUTION **E**

We have

$$40\% \text{ of } 50\% \text{ of } x = 0.4 \times 0.5 \times x = 0.2x,$$

and

$$20\% \text{ of } 30\% \text{ of } y = 0.2 \times 0.3 \times y = 0.06y.$$

Since these are equal

$$0.06y = 0.2x.$$

Hence

$$6y = 20x.$$

So

$$y = \frac{20}{6}x = \frac{10}{3}x.$$

FOR INVESTIGATION

12.1 Andrew calculates that 40% of 50% of 60% of x is equal to 70% of 80% of 90% of y, where $x \neq 0$. Find the ratio $x : y$.

13. What is the remainder when $12\,345 \times 54\,321$ is divided by 9?

A 0 B 1 C 2 D 3 E 4

SOLUTION **A**

The sum of the digits of the integer 12345 is $1 + 2 + 3 + 4 + 5 = 15$. Because 15 is a multiple of 3, we can deduce that 12345 is a multiple of 3. [See Problem 13.1.] Similarly, 54321 is a multiple of 3. Therefore 12345×54321 is a multiple of 9.

It follows that the remainder when 12345×54321 is divided by 9 is 0.

FOR INVESTIGATION

13.1 (a) In the solution above we have used the fact that a criterion for whether an integer is divisible by 3 is that the sum of its digits is a multiple of 3.

Explain why this is correct.

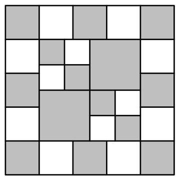
(b) Show, more generally, that the remainder when a positive integer is divided by 3 is equal to the remainder when the sum of its digits is divided by 3.

13.2 Show that the remainder when a positive integer is divided by 9 is equal to the remainder when the sum of its digits is divided by 9.

13.3 What is the remainder when the number $12\,345\,678\,987\,654\,321$ is divided by 9?

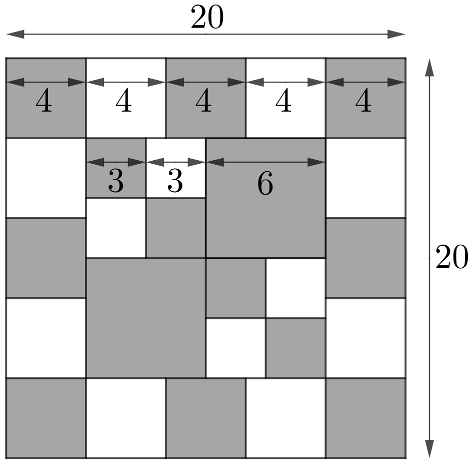
14. The diagram shows a large square divided into squares of three different sizes.
What percentage of the large square is shaded?

A 61% B 59% C 57% D 55% E 53%



Solution **B**

We see from the diagram that the width of four of the smallest shaded squares is equal to the width of three of the medium sized shaded squares.



Therefore, to avoid fractions, we suppose that the smallest shaded squares have side length 3.

Then the medium-sized shaded squares have side length 4.

It follows that the largest shaded squares have side length 6 and the large square has side length 20.

There are four of the smallest shaded squares, eight of the medium-sized shaded squares and two of the largest shaded squares.

It follows that the shaded area of the large square is

$$4(3 \times 3) + 8(4 \times 4) + 2(6 \times 6) = 36 + 128 + 72 = 236.$$

Therefore the fraction of the large square that is shaded is

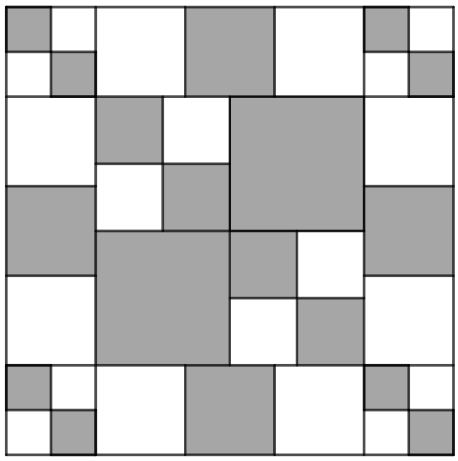
$$\frac{236}{20 \times 20} = \frac{236}{400} = \frac{59}{100}.$$

Hence 59% of the large square is shaded.

For investigation

14.1 The diagram on the right shows a large square divided into squares of four different sizes.

What percentage of the large square is shaded?



15. Patrick drives from P to Q at an average speed of 40 mph. His drive back from Q to P is at an average speed of 45 mph and takes two minutes less.
How far, in miles, is it from P to Q?

A 1.5 B 6 C 9 D 12 E 15

Solution **D**

Let x be the number of miles from P to Q.

It takes Patrick $\frac{x}{40}$ hours to drive from P to Q at 40 mph, and $\frac{x}{45}$ hours to drive from Q to P at 45 mph.

The difference in the time taken for these two drives is two minutes, that is, $\frac{1}{30}$ hours. Therefore

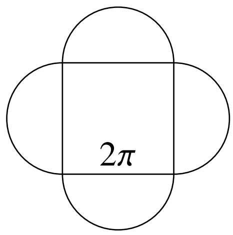
$$\frac{1}{30} = \frac{x}{40} - \frac{x}{45} = \left(\frac{1}{40} - \frac{1}{45}\right)x = \left(\frac{45-40}{40 \times 45}\right)x == \left(\frac{5}{40 \times 45}\right)x = \frac{1}{360}x.$$

It follows that

$$x = \frac{360}{30} = 12.$$

Therefore it is 12 miles from P to Q.

16. A semicircle is drawn on each side of a square, as shown.
The square has sides of length 2π.
What is the area of the resulting shape?



A $2\pi^2(\pi + 1)$ B $\pi^2(\pi + 2)$ C $2\pi^2(2\pi + 1)$
D $\pi^2(\pi + 4)$ E $2\pi^2(\pi + 2)$

Solution **E**

The square with sides of length 2π has area $(2\pi)^2$, that is $4\pi^2$.

Because the sides of the square have length 2π, the radius of each semicircle is π. Hence the area of each semicircle is $\frac{1}{2}(\pi(\pi^2))$. Therefore the total area of the four semicircles is $4 \times (\frac{1}{2}\pi(\pi^2)) = 2\pi^3$.

Therefore the area of the shape is $2\pi^3 + 4\pi^2 = 2\pi^2(\pi + 2)$.

For investigation

16.1 Show that it is possible to draw the shape of this question without taking your pencil off the paper, and without drawing over the same line twice.

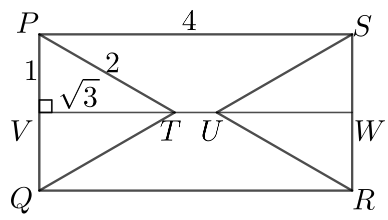
16.2 What is the length of the route taken by the pencil in Problem 16.1?

17. In the rectangle $PQRS$, the side PQ is of length 2 and the side QR is of length 4. Points T and U lie inside the rectangle so that PQT and RSU are equilateral triangles. What is the area of the quadrilateral $QRUT$?

A $\frac{6-\sqrt{3}}{2}$ B $\frac{8}{3}$ C $4-2\sqrt{3}$ D $4-\sqrt{3}$ E 3

SOLUTION **D**

The quadrilaterals $QRUT$ and $PSUT$ are congruent [see Problem 17.1] and therefore the area of $QRUT$ is half the area of the hexagon $PTQRUS$.



The area of the hexagon $PTQRUS$ equals the area of the rectangle $PQRS$ less the area of the triangles PQT and RSU. [For an alternative approach see Problem 17.5.]

The rectangle $PQRS$ has area $2 \times 4 = 8$.

Let V be the midpoint of PQ and W be the midpoint of SR. Then $PV = QV = 1$.

Because the triangle PQT is equilateral, $PT = PQ = 2$.

The triangles PVT and QVT are congruent [see Problem 17.2] and therefore $\angle PVT = \angle QVT = 90°$.

Therefore, by Pythagoras' Theorem applied to the right-angled triangle PVT, we deduce that $PV^2 + VT^2 = PT^2$, that is, $1^2 + VT^2 = 2^2$. It follows that $VT^2 = 2^2 - 1^2 = 3$, and therefore $VT = \sqrt{3}$.

We can now deduce, using the formula $\frac{1}{2}$(base $\times$ height) for the area of a triangle, that the area of the triangle PQT is given by $\frac{1}{2}(PQ \times VT) = \frac{1}{2}(2 \times \sqrt{3}) = \sqrt{3}$.

Similarly, the triangle RSU has area $\sqrt{3}$.

Therefore the hexagon $PTQRUS$ has area $8 - 2\sqrt{3}$.

Hence the area of the quadrilateral $QRUT$ is $\frac{1}{2}(8 - 2\sqrt{3}) = 4 - \sqrt{3}$.

FOR INVESTIGATION

17.1 Prove that the quadrilaterals $QRUT$ and $PSUT$ are congruent.

17.2 (a) Prove that the triangles QVT and PVT are congruent.

(b) Deduce that $\angle PVT = \angle QVT = 90°$.

17.3 Find a formula for the area of an equilateral triangle with side lengths s.

17.4 What is the side length of an equilateral triangle whose area is 1?

17.5 (a) Show that TU is parallel to QR and hence that $QRUT$ is a trapezium.

(b) Show that the area of a trapezium is $\frac{1}{2}h(a+b)$, where a and b are the lengths of the parallel sides, and h is their distance apart.

(c) Find the length of TU and use this to calculate directly the area of the trapezium $QRUT$.

18. Which of these is closest in size to 1?

A $0.\dot{9}\dot{5}$ B $1.\dot{0}\dot{5}$ C $0.\dot{9}6\dot{0}$ D $1.\dot{0}4\dot{0}$ E $0.9\dot{5}$

SOLUTION **C**

Note: A good first move is to arrange the recurring decimals given in this question in order of size.

To do this it may help to think about, for example, the first 6 decimal places of these numbers.

We see that

$$\begin{aligned}
0.\dot{9}\dot{5} &= 0.959595\ldots\\
1.\dot{0}\dot{5} &= 1.050505\ldots\\
0.\dot{9}6\dot{0} &= 0.960960\ldots\\
1.\dot{0}4\dot{0} &= 1.040040\ldots\\
0.9\dot{5} &= 0.955555\ldots
\end{aligned}$$

We see from this that

$$0.9\dot{5} < 0.\dot{9}\dot{5} < 0.\dot{9}6\dot{0} < 1 < 1.\dot{0}4\dot{0} < 1.\dot{0}\dot{5}.$$

It follows that $0.\dot{9}6\dot{0}$ is closer 1 to than either $0.9\dot{5}$ or $0.\dot{9}\dot{5}$, and that $1.\dot{0}4\dot{0}$ is closer to 1 than $1.\dot{0}\dot{5}$.

Therefore it remains only to decide which of $0.\dot{9}6\dot{0}$ and $1.\dot{0}4\dot{0}$ is closer to 1.

We also see that

$$1.\dot{0}4\dot{0} - 1 = 0.040040\ldots$$

and that

$$1 - 0.\dot{9}6\dot{0} = 0.039039\ldots\ .$$

Since $0.039039\cdots < 0.040040\ldots$ we see that $0.\dot{9}6\dot{0}$ is closer to 1 than is $1.\dot{0}4\dot{0}$.

We conclude that the closest to 1 of the five recurring decimals given in this question is $0.\dot{9}6\dot{0}$.

FOR INVESTIGATION

18.1 (a) Express the decimals $0.\dot{9}6\dot{0}$ and $1.\dot{0}4\dot{0}$ as fractions.

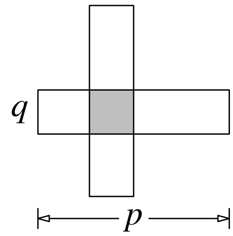
(b) Deduce that $1 - 0.\dot{9}6\dot{0} < 1.\dot{0}4\dot{0} - 1$.

18.2 Express the recurring decimals $0.9\dot{5}$, $0.\dot{9}\dot{5}$ and $1.\dot{0}\dot{5}$ as fractions.

18.3 Arrange the recurring decimals $0.\dot{1}2\dot{3}$, $0.1\dot{2}\dot{3}$ and $0.12\dot{3}$ in order of size.

18.4 Let r and s be two digits. Express the recurring decimal $0.\dot{r}\dot{s}$ as a fraction.

19. The diagram shows two overlapping rectangles, each measuring p by q. The area of overlap is exactly one-quarter of the total area of the figure.
What is the ratio $p : q$?



A 5 : 2 B 4 : 1 C 3 : 1 D 2 : 1 E 3 : 2

SOLUTION **A**

The area of each rectangle is pq. The overlap is a q by q square with area q^2.

The area of the figure is the sum of the areas of the two rectangles less the area of the overlap. Thus the area of the figure is $2pq - q^2$.

Because the area of the overlap is one-quarter of the area of the figure, we have

$$q^2 = \tfrac{1}{4}(2pq - q^2).$$

This equation may be rearranged as

$$4q^2 = 2pq - q^2.$$

This gives $5q^2 = 2pq$. Hence, as $q \neq 0$, we can deduce that $5q = 2p$, therefore $\dfrac{p}{q} = \dfrac{5}{2}$.

It follows that $p : q = 5 : 2$.

FOR INVESTIGATION

19.1 Suppose that the area of the overlap is exactly one-fifth of the total area of the figure.

What is the ratio $p : q$ in this case?

19.2 (a) Find a formula, in terms of k, for the ratio $p : q$ in the case where ratio of the area of overlap to the total area of the figure is $1 : k$.

(b) Check that your formula is compatible with your answer to Question 19 and with your answer to Problem 19.1.

20. Two straight lines have equations $y = px + 4$ and $py = qx - 7$, where p and q are constants.
The two lines meet at the point (3, 1).
What is the value of q?

A 1 B 2 C 3 D 4 E 5

SOLUTION **B**

Because the lines meet at the point $(3, 1)$, the point $(3, 1)$ lies on both lines.

This means that both equations are satisfied when we put $x = 3$ and $y = 1$. Therefore

$$1 = 3p + 4 \qquad (1)$$

and

$$p = 3q - 7. \qquad (2)$$

From (1) we have

$$3p = -3.$$

Therefore

$$p = -1.$$

Substituting this value in (2) gives

$$-1 = 3q - 7.$$

Hence

$$3q = 6.$$

We can now deduce that

$$q = 2.$$

FOR INVESTIGATION

20.1 The straight lines with equations $y = px + 4$ and $py = qx - 7$, where $p \neq 0$, are parallel.

The line with equation $y = px + 4$ meets the x-axis at the point with coordinates $(2, 0)$.

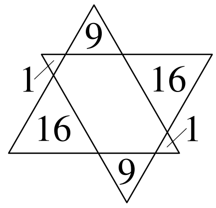
What are the coordinates of the point where the line with equation $py = qx - 7$ meets the x-axis?

20.2 The straight lines with equations $y = px + 4$ and $py = qx - 7$, where $p \neq 0$, are perpendicular.

The line with equation $y = px + 4$ meets the y-axis at the point with coordinates $(0, 1)$.

What are the coordinates of the point where the line with equation $py = qx - 7$ meets the y-axis?

21. The diagram shows two congruent equilateral triangles whose overlap is a hexagon. The areas of the smaller triangles, which are also equilateral, are 1, 1, 9, 9, 16 and 16, as shown.
What is the area of the inner hexagon?

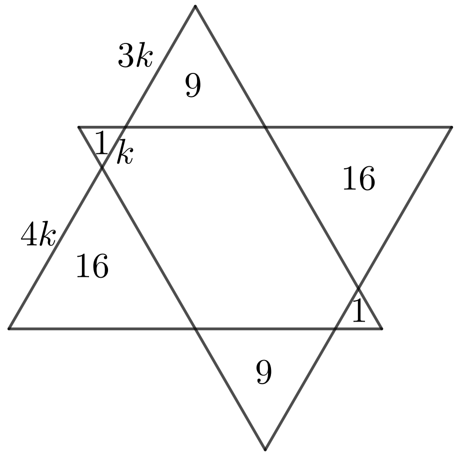


A 68 B 58 C 48 D 38 E 28

Solution **D**

The ratio of the areas of similar triangles equals the ratio of the squares of their side lengths. [You are asked to prove this in Problem 21.1.]

Therefore the ratio of the side lengths of the equilateral triangles with areas 1 and 9 is 1 : 3. Thus, if we let k be the side length of the equilateral triangles with area 1, it follows that the equilateral triangles with area 9 have side length $3k$.



Similarly, the equilateral triangles with area 16 have side length $4k$.

We now see from the diagram that the large equilateral triangles have side length $8k$. It follows that these triangles have area 8^2, that is 64.

The area of the inner hexagon equals the area of one of the large equilateral triangles, less the areas of one triangle with area 1, one with area 9 and one with area 16.

Therefore the area of the inner hexagon is

$$64 - 1 - 9 - 16 = 38.$$

22. What is the result when we simplify the expression $\left(1+\frac{1}{x}\right)\left(1-\frac{2}{x+1}\right)\left(1+\frac{2}{x-1}\right)$?

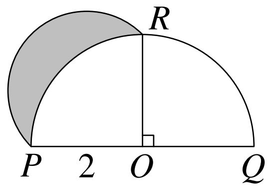
A 1 B $\frac{1}{x(x+1)}$ C $\frac{1}{(x+1)(x-1)}$ D $\frac{1}{x(x+1)(x-1)}$ E $\frac{x+1}{x}$

Solution **E**

We have

$$\begin{aligned}\left(1+\frac{1}{x}\right)\left(1-\frac{2}{x+1}\right)\left(1+\frac{2}{x-1}\right) &= \left(\frac{x+1}{x}\right)\left(\frac{(x+1)-2}{x+1}\right)\left(\frac{(x-1)+2}{x-1}\right)\\ &= \left(\frac{x+1}{x}\right)\left(\frac{x-1}{x+1}\right)\left(\frac{x+1}{x-1}\right)\\ &= \frac{x+1}{x}.\end{aligned}$$

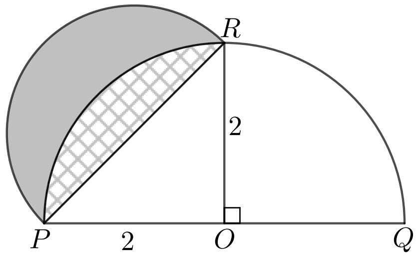
23. The diagram shows a semicircle with centre O and radius 2 and a semicircular arc with diameter PR. Angle POR is a right angle.
What is the area of the shaded region?



A $\pi - 2$ B 2 C π D 3 E $2\pi - 2$

SOLUTION **B**

The area of the shaded region is that of the semicircle with diameter PR less the area of the hatched region.



By Pythagoras' Theorem applied to the right-angled triangle POR, we have $PR^2 = PO^2 + OR^2 = 2^2 + 2^2 = 8$. Therefore $PR = \sqrt{8} = 2\sqrt{2}$.

Thus the radius of the semicircle is $\sqrt{2}$. Hence its area is $\frac{1}{2}(\pi(\sqrt{2}^2)) = \pi$.

The hatched area is the area of the quarter circle with centre O and radius 2 that goes through P and R, less the area of the triangle POR. Therefore the hatched area is $\frac{1}{4}(\pi(2^2)) - \frac{1}{2}(2 \times 2) = \pi - 2$.

It follows that the area of the shaded region is $\pi - (\pi - 2) = 2$.

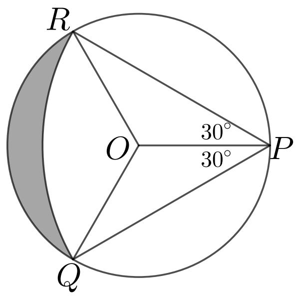
FOR INVESTIGATION

23.1 O is the centre of a circle of radius 1.

P, Q and R are points on the circle such that $\angle OPQ = \angle OPR = 30°$.

The shaded region is the region between the circle centre O and the circular arc with centre P that goes through Q and R.

What is the area of the shaded region?



COMMENTARY

The shaded shape of this question is called a *lune*, as also is the shaded shape in Problem 23.1.

Note that the shape of the question is bounded by arcs of circles. These circles have areas 2π and 4π, which are both irrational numbers. However the area of the lune is the integer 2.

The method given here for finding the area of the lune is attributed to the Greek mathematician Hippocrates of Chios (470BC-410BC). He should not be confused with the physician Hippocrates of Kos who gave his name to the medical *Hippocratic oath*.

24. Sam writes on a white board the positive integers from 1 to 6 inclusive, once each. She then writes p additional fives and q sevens on the board. The mean of all the numbers on the board is then 5.3.
What is the smallest possible value of q?

A 7 B 9 C 11 D 13 E 15

Solution **B**

Sam writes $6 + p + q$ numbers on the white board.

The total of these numbers is $1 + 2 + 3 + 4 + 5 + 6 + 5p + 7q = 21 + 5p + 7q$.

The mean of all the numbers is 5.3. Therefore

$$\frac{21 + 5p + 7q}{6 + p + q} = 5.3 = \frac{53}{10}.$$

Therefore

$$10(21 + 5p + 7q) = 53(6 + p + q).$$

Hence

$$210 + 50p + 70q = 318 + 53p + 53q.$$

It follows that

$$3p = 17q - 108.$$

and hence

$$p = \frac{17}{3}q - 36.$$

Since p is an integer, q must be multiple of 3.

Therefore, because p cannot be negative, q is the least multiple of 3 such that $\frac{17}{3}q \geq 36$, that is such that $17q \geq 108$.

We have $17 \times 6 = 102 < 108$, but $17 \times 9 = 153 > 108$.

We deduce that the smallest possible value of q is 9.

For investigation

24.1 Sam writes on a white board the positive integers from 1 to 10 inclusive, once each. She then writes p additional nines and q additional tens on the board. The mean of all the numbers on the board is 7.5.

What is the largest possible value of p?

25. Thomas has constant speeds for both running and walking. When a down-escalator is moving, Thomas can run down it in 15 seconds or walk down it in 30 seconds. One day, when the escalator was broken (and stationary), it took Thomas 20 seconds to run down it.
How long, in seconds, would it take Thomas to walk down the broken escalator?

A 30 B 40 C 45 D 50 E 60

Solution **E**

Let Thomas' speed when running be r metres per second and his speed when walking be w metres per second. Suppose that the escalator moves at e metres per second when it is working.

Let the length of the escalator be M metres.

Because it takes Thomas 20 seconds to run down the broken escalator,

$$M = 20r. \qquad (1)$$

When the escalator is moving down at e metres per second, Thomas' downwards speed when he is running is $e + r$ metres per second. It takes him $\frac{M}{e+r}$ seconds to get down the escalator at this speed. Because $M = 20r$ and it takes him 15 seconds to get down the escalator at this speed,

$$\frac{20r}{e+r} = 15. \qquad (2)$$

Hence $20r = 15(e + r)$ and so $5r = 15e$. Therefore

$$e = \tfrac{1}{3}r. \qquad (3)$$

Similarly, because it takes Thomas 30 seconds to walk down the escalator when it is working,

$$\frac{20r}{e+w} = 30. \qquad (4)$$

Hence $20r = 30(e + w)$, and so, $2r = 3(e + w)$. Hence, by (3), $6e = 3e + 3w$, and so, $w = e$. Therefore, using (3) again,

$$w = \tfrac{1}{3}r. \qquad (5)$$

When Thomas walks down the broken escalator his speed is w metres per second. Therefore it takes him $\frac{M}{w}$ seconds to walk down the broken escalator. From (1) and (5) it follows that

$$\frac{M}{w} = \frac{20r}{\frac{1}{3}r} = 60.$$

We conclude that it would take Thomas 60 seconds to walk down the broken escalator.

For investigation

25.1 It takes Thomas 40 seconds to run *up* the broken down-escalator. How long would it take Thomas to run up the down-escalator when it is working?