United Kingdom
Mathematics Trust

Intermediate Mathematical Olympiad

Hamilton paper

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

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1. Arun and Disha have some numbered discs to share out between them. They want to end up with one pile each, not necessarily of the same size, where Arun's pile contains exactly one disc numbered with a multiple of 2 and Disha's pile contains exactly one disc numbered with a multiple of 3. For each case below, either count the number of ways of sharing the discs, or explain why it is impossible to share them in this way.

(a) They start with ten discs numbered from 1 to 10.

(b) They start with twenty discs numbered from 1 to 20.

Solution

(a) Consider the disc numbered 6. Assume for the moment that it is in Arun's pile. Then all the other discs numbered with a multiple of 2 must be in Disha's pile.

Disha's pile must also contain one of the two remaining discs numbered with a multiple of 3 (with the other in Arun's pile), so there are ${}^2C_1 = 2$ ways to arrange the remaining discs numbered with a multiple of 3.

Once the discs containing multiples of 3 have been allocated to a pile, there are three remaining discs (those numbered 1, 5 and 7), each of which could go in either of the two piles, so there are $2^3 = 8$ ways this can be done.

So there are $2 \times 8 = 16$ ways of distributing the discs if the disc numbered 6 is in Arun's pile.

Now assume that the disc numbered 6 is in Disha's pile. Then all the other discs numbered with a multiple of 3 must be in Arun's pile.

Arun's pile must also contain one of the four remaining discs numbered with a multiple of 2 (with the other three all in Disha's pile), so there are ${}^4C_1 = 4$ ways to arrange the remaining discs numbered with a multiple of 2.

Once the discs containing multiples of 2 have been allocated to a pile, there are again three remaining discs (1, 5 and 7), and as before there are 8 ways these can be allocated to the piles.

So there are $4 \times 8 = 32$ ways of distributing the discs if the disc numbered 6 is in Disha's pile.

Hence there are $16 + 32 = 48$ ways they can share the discs out.

(b) Consider the three discs numbered 6, 12 and 18, which are multiples of both 2 and 3. Since there are three of them, when the discs are shared out, one of Arun's and Disha's pile must contain (at least) two of them. But that is against the rules for how Arun and Disha want to share the discs out, so it is impossible to share them out.

2. In the UK, 1p, 2p and 5p coins have thicknesses of 1.6 mm, 2.05 mm and 1.75 mm respectively.

Using only 1p and 5p coins, Joe builds the shortest (non-empty) stack he can whose height in millimetres is equal to its value in pence. Penny does the same but using only 2p and 5p coins.

Whose stack is more valuable?

Solution

Say Joe has a 1p coins and b 5p coins. Then he wants the minimal a and b such that

$$a + 5b = 1.6a + 1.75b.$$

This simplifies to

$$12a = 65b.$$

Since a and b are integers, and 12 and 65 have no factors in common (other than 1), a must be a multiple of 65 and b must be a multiple of 12. Clearly the smallest (positive) a and b which satisfy this equation are $a = 65$ and $b = 12$. So Joe's stack comprises sixty-five 1p coins and twelve 5p coins, giving it a value of 125p.

Now say Penny has c 2p coins and d 5p coins. Then similarly, she wants the minimal c and d such that

$$2c + 5d = 2.05c + 1.75d,$$

which simplifies to

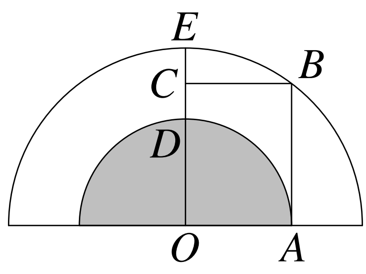
$$c = 65d.$$

The smallest positive c and d which satisfy this equation are $c = 65$ and $d = 1$. So Penny's stack comprises sixty-five 2p coins and one 5p coin, giving it a value of 135p.

Hence Penny's stack is more valuable.

3. The diagram shows two semicircles with a common centre O and a rectangle $OABC$. The line through O and C meets the small semicircle at D and the large semicircle at E. The lengths CD and CE are equal.

What fraction of the large semicircle is shaded?



Solution

Say $OA = r$ and $OB = R$. Then $OC = \frac{R+r}{2}$.

By Pythagoras' Theorem,

$$\begin{aligned} r^2 + (\frac{R+r}{2})^2 &= R^2 \\ 3R^2 - 2Rr - 5r^2 &= 0 \\ (3R - 5r)(R + r) &= 0 \end{aligned}$$

So (discounting $r = -R$, which is not practicable) $r = \frac{3}{5}R$.

Hence the fraction of the larger semicircle that is shaded is $(\frac{3}{5})^2 = \frac{9}{25}$.

4. Piercarlo chooses n integers from 1 to 1000 inclusive. None of his integers is prime, and no two of them share a factor greater than 1.

What is the greatest possible value of n?

Solution

First, note that $31^2 = 961$ and $37^2 = 1369$, so the largest prime less than $\sqrt{1000}$ is 31.

So any non-prime number less than 1000 (excepting 1) must have at least one prime factor which is less than or equal to 31 (otherwise the number would be at least 37^2, which is larger than 1000).

We are told that all pairs of Piercarlo's share no factors greater than one; this clearly must include prime factors, and in particular the primes from 2 to 31 inclusive. Hence each of the primes up to 31 can only appear in the factorisation of one of his numbers.

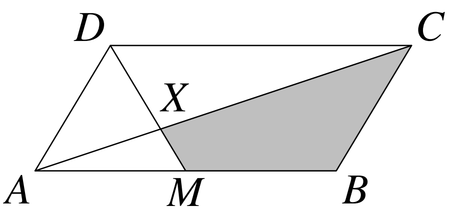
So to maximise n, he will need to choose numbers with as few of these prime factors in their prime factorisation as possible. Indeed, he should aim to choose a set of numbers where each number he chooses has no more than one prime factor less than or equal to 31. Since there are 11 of these primes, the maximum value of n is 12 (since he can also pick 1).

It is easy to check that he can indeed pick 12 non-prime numbers less than 1000 – for example by picking 1 together with the squares of all the primes up to 31:

$$\{1, 4, 9, 25, 49, 121, 169, 289, 361, 529, 841, 961\}.$$

5. In the diagram, $ABCD$ is a parallelogram, M is the midpoint of AB and X is the point of intersection of AC and MD.

What is the ratio of the area of $MBCX$ to the area of $ABCD$?



Solution

Solution 1

Say that the area of triangle XAM is 1 square unit. Note that $\angle XMA = \angle XDC$ and $\angle XAM = \angle XCD$ (since they are alternate angles between parallel lines), so triangles XCD and XAM are similar. Since $DC = 2AM$, the area of XCD is $1 \times 2^2 = 4$ square units.

Say the area of triangle DAX is x square units. Then, since the area of ACD is twice the area of AMD (they are triangles with equal heights and one base half that of the other), we have $2 \times (1 + x) = 4 + x$, which gives $x = 2$ square units.

The area of $ABCD$ is twice that of ACD, which is 12 square units.

The area of $MBCX$ is $12 - (4 + 2 + 1) = 5$ square units.

So the desired ratio is 5:12.

Solution 2

Note that $\angle XMA = \angle XDC$ and $\angle XAM = \angle XCD$ (since they are alternate angles between parallel lines), so triangles XCD and XAM are similar. Since $DC = 2AM$, the perpendicular from DC to X is twice the perpendicular from AM to X.

Hence the perpendicular height of the triangle AMX is one third of the perpendicular height of the parallelogram. Hence the area of AMX is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{3} = \frac{1}{12}$ of the area of the parallelogram.

The area of triangle ABC is clearly half of the area of the parallelogram, so the area of $MBCX$ is $\frac{1}{2} - \frac{1}{12} = \frac{5}{12}$ of the area of the whole parallelogram.

Hence the desired ratio is 5:12.

6. We write $\lfloor x \rfloor$ to represent the largest integer less than or equal to x. So, for example, $\lfloor 1.7 \rfloor = 1$, $\lfloor 2 \rfloor = 2$, $\lfloor \pi \rfloor = 3$ and $\lfloor -0.4 \rfloor = -1$.

Find all real values of x such that $\lfloor 3x + 4 \rfloor = \lfloor 5x - 1 \rfloor$.

SOLUTION

Solution 1

First note that $x - 1 < \lfloor x \rfloor \leq x$ (*).

Also note that, for any integer n, $\lfloor x + n \rfloor = \lfloor x \rfloor + n$.

So the equation $\lfloor 3x + 4 \rfloor = \lfloor 5x - 1 \rfloor$ can be rearranged to give

$$\lfloor 5x \rfloor - \lfloor 3x \rfloor = 5.$$

Now, by (*), $5x - (3x - 1) > 5$, which simplifies to $x > 2$, and $(5x - 1) - 3x < 5$, which simplifies to $x < 3$.

Given $2 < x < 3$, the value of $\lfloor 3x \rfloor$ will only change as x changes value above/below $2\frac{1}{3}$ and $2\frac{2}{3}$.

Similarly, the value of $\lfloor 5x \rfloor$ will only change as x changes value above/below $2\frac{1}{5}$, $2\frac{2}{5}$, $2\frac{3}{5}$ and $2\frac{4}{5}$.

So now we just need to consider the regions between 2 and 3 created by these values:

x	$\lfloor 3x + 4 \rfloor$	$\lfloor 5x - 1 \rfloor$
$2 < x < 2\frac{1}{5}$	10	9
$2\frac{1}{5} \leq x < 2\frac{1}{3}$	10	10
$2\frac{1}{3} \leq x < 2\frac{2}{5}$	11	10
$2\frac{2}{5} \leq x < 2\frac{3}{5}$	11	11
$2\frac{3}{5} \leq x < 2\frac{2}{3}$	11	12
$2\frac{2}{3} \leq x < 2\frac{4}{5}$	12	12
$2\frac{4}{5} \leq x < 3$	12	13

So the set of values for which the equation holds is {x such that $2\frac{1}{5} \leq x < 2\frac{1}{3}$ or $2\frac{2}{5} \leq x < 2\frac{3}{5}$ or $2\frac{2}{3} \leq x < 2\frac{4}{5}$}.

Solution 2

First note that $x - 1 < \lfloor x \rfloor \leq x \qquad (*)$.

Say that $\lfloor 3x + 4 \rfloor = k$ for some integer k. Then from (*) we have $3x + 4 \geq k$ and $3x + 3 < k$. These can be rearranged and combined to give

$$\frac{k-4}{3} \leq x < \frac{k-3}{3}.$$

Now we also have $\lfloor 5x - 1 \rfloor = k$, so again from (*) we have $5x - 2 < k$ and $5x - 1 \geq k$, which can be rearranged and combined to give

$$\frac{k+1}{5} \leq x < \frac{k+2}{5}.$$

The solutions to the equation will be where these two regions of the number line overlap.

First, we need the two regions on the number line to overlap at all, so both:

$$\frac{k+1}{5} < \frac{k-3}{3}, \text{ which gives } k > 9,$$

$$\text{and } \frac{k-4}{3} < \frac{k+2}{5}, \text{ which gives } k < 13.$$

We can check the three remaining values of k:

When $k = 10$, the overlap is $2\frac{1}{5} \leq x < 2\frac{1}{3}$.

When $k = 11$, the overlap is $2\frac{2}{5} \leq x < 2\frac{3}{5}$.

And when $k = 12$, the overlap is $2\frac{2}{3} \leq x < 2\frac{4}{5}$.

These three regions of overlap are exactly the regions where the equation is satisfied.