4. (a) $\{a_1, a_2, \ldots, a_5\}$ is a permutation of $\{1, 3, 4, 5, 8\}$ (that is, $a_1, a_2, \ldots, a_5$ are equal to 1, 3, 4, 5 and 8 in some order) and $\{b_1, b_2, \ldots, b_5\}$ is a permutation of $\{2, 5, 9, 11, 14\}$. What are the maximum and minimum possible values of

$$a_1b_1 + a_2b_2 + \cdots + a_5b_5 \ ?$$

Can you give a good justification for these?

(b) a, b and c are positive real numbers. Show, using your ideas from part (a), that

$$a^3 + b^3 + c^3 \geqslant a^2b + b^2c + c^2a.$$

(a) **Discussion:** There seem to be two choices: the values of the a_i and the values of the b_j. We may as well fix the a_i and consider which permutation of the b_j maximises or minimises $a_1b_1+a_2b_2+\cdots+a_5b_5$. We set $a_1 = 1$, $a_2 = 3$, $a_3 = 4$, $a_4 = 5$ and $a_5 = 8$ and are looking for the maximum and minimum possible values of

$$b_1 + 3b_2 + 4b_3 + 5b_4 + 8b_5, \tag{†}$$

where $\{b_1, b_2, \ldots, b_5\}$ is a permutation of $\{2, 5, 9, 11, 14\}$. To make (†) largest, it would seem sensible to pair the larger b_j with the larger a_i (so take $b_1 = 2, b_2 = 5, b_3 = 9, b_4 = 11$ and $b_5 = 14$) – this would give a maximum value of $1 \times 2 + 3 \times 5 + 4 \times 9 + 5 \times 11 + 8 \times 14 = 220$.

The fact that the maximum is attained when the a_i and b_j are ordered in the same way (largest paired with largest, second largest with second largest, ..., smallest with smallest) seems intuitive and certainly for some this would be enough justification. Here we will give a more detailed proof which throws up some useful ideas and techniques.

Proving this in one go seems fiddly with so many variables about. However doing it bit-by-bit seems more manageable: we will swap pairs of b_i until they are in the right order. We need to show that (†) does not decrease with every swap and this should not be too bad as each swap affects only two of the terms.

In a similar vein, to make (†) smallest, it would seem sensible for the larger b_j to be paired with the smaller a_i (so take $b_1 = 14, b_2 = 11, b_3 = 9, b_4 = 5$ and $b_5 = 2$) – this would give a minimum value of $1 \times 14 + 3 \times 11 + 4 \times 9 + 5 \times 5 + 8 \times 2 = 124$.

Solution: We may take $a_1, a_2, \ldots, a_5$ so that $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_5$. We will show that $a_1b_1 + a_2b_2 + \cdots + a_5b_5$ is maximised when $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_5$. Suppose that b_j are not in increasing order: we have $i < j$ with $b_i = x > y = b_j$.

Swapping b_i and b_j (so that b_i becomes y and b_j becomes x) increases $a_1b_1 + a_2b_2 + \cdots + a_5b_5$ by

$$(a_iy + a_jx) - (a_ix + a_jy). \tag{$*$}$$

Provided we can show this last quantity is non-negative we are done: we repeatedly swap pairs of b_j until they are in increasing order, all the while not decreasing the value of $a_1b_1 + a_2b_2 + \cdots + a_5b_5$. Fortunately ($*$) factorises as $(a_j - a_i)(x - y)$ which is non-negative as each bracket is (note our choice of ordering the a_i means that $a_i \leqslant a_j$). Hence we are done and the maximum value is 220.

The proof for the minimum is very similar: this time we show that $a_1b_1+a_2b_2+\cdots+a_5b_5$ is minimsed when $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_5$. We do this by swapping the b_j until they are in this order and checking that each swap does not increase $a_1b_1 + a_2b_2 + \cdots + a_5b_5$. The minimum value is 124.

Remark: The factorisation $ax + by - ay - bx = (a - b)(x - y)$ is a very useful one (when $a = x$ and b and y are given numbers it is the familiar factorisation of a quadratic). As an example, can you find all integers m and n such that $mn + 6m - 4n = 27$ (there are 4 solutions).

The fact that given two lists of numbers, the largest sum of pairwise products is obtained when the two lists are ordered in the same way (and the smallest sum is obtained when the two lists are ordered in the opposite way) is called the *rearrangement inequality*.

(b) **Discussion:** By writing $a^3 + b^3 + c^3$ as $a^2 \times a + b^2 \times b + c^2 \times c$ we see that both sides are of the form $a_1b_1 + a_2b_2 + a_3b_3$ where $\{a_1, a_2, a_3\}$ is a permutation of $\{a^2, b^2, c^2\}$ and $\{b_1, b_2, b_3\}$ is a permutation of $\{a, b, c\}$.

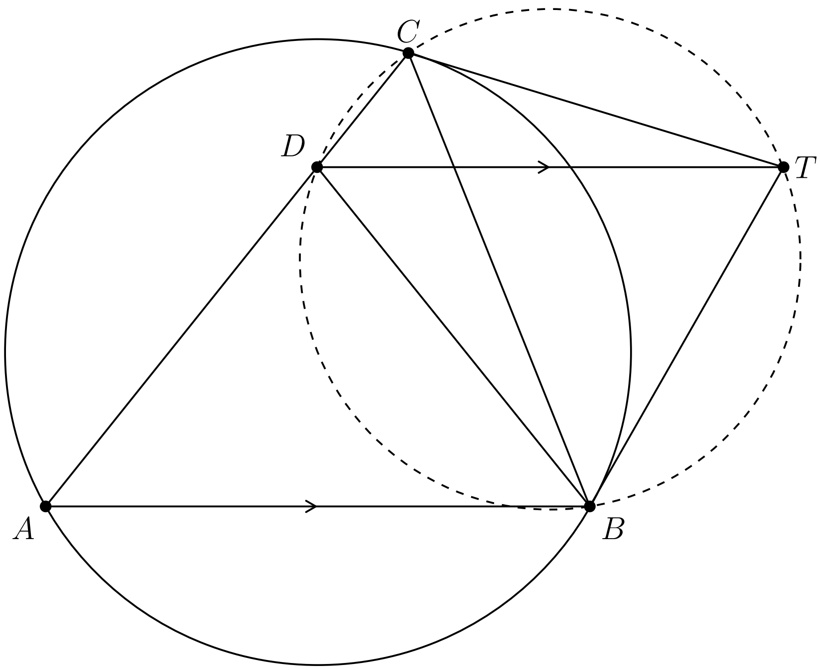
Solution: a, b and c are positive so the lists a, b, c and a^2, b^2, c^2 are ordered in the same way (if a is largest and b is smallest then a^2 is largest and b^2 is smallest and so on).

Let $\{a_1, a_2, a_3\}$ be a permutation of $\{a^2, b^2, c^2\}$ and $\{b_1, b_2, b_3\}$ be a permutation of $\{a, b, c\}$. $a_1b_1 + a_2b_2 + a_3b_3$ takes its largest value when a_1, a_2, a_3 is ordered in the same was as b_1, b_2, b_3 so:

$$a^3 + b^3 + c^3 = a^2 \times a + b^2 \times b + c^2 \times c \geqslant a^2 \times b + b^2 \times c + c^2 \times a = a^2b + b^2c + c^2a.$$

2. ABC is a triangle and S is the circle passing through A, B and C. The tangents to S at B and C meet at T. Let the line through T parallel to AB meet AC at D. Prove that points B, C, D and T lie on a circle and that $AD = BD$.

Discussion: There are a few ways to prove that four points lie on a circle or that two lengths are equal (it is a useful exercise to come up with a list of all the ways you know). The parallel lines and tangents (letting us use the alternate segment theorem) provide us with many equal angles so it is easiest to work with angles. $AD = BD$ has an equivalent angle condition ($A\hat{B}D = D\hat{A}B$) and the converse of equal angles in the same segment is a common way of prove four points form a cyclic quadrilateral so chasing a few angles looks very promising.



Solution: We first show that $BTCD$ is cyclic. Using parallel lines followed by the alternate segment theorem, $C\hat{D}T = C\hat{A}B = C\hat{B}T$, so, by the converse of equal angles in the same segment, $BTCD$ is cyclic.

Now, using the parallel lines, $A\hat{B}D = T\hat{D}B$. $BTCD$ is cyclic so $T\hat{D}B = T\hat{C}B$. Finally, by the alternate segment theorem, $T\hat{C}B = C\hat{A}B = D\hat{A}B$. Thus $A\hat{B}D = D\hat{A}B$ and so triangle BDA is isosceles with $AD = BD$.

Remarks: People often draw the centres of circles and radii on geometry diagrams. Unless you have a good reason to do so this can clutter up the diagram and you may end up wasting time and getting bogged down, for example by chasing a few angles only to find you have effectively just proved the alternate segment theorem. Many circle theorems (alternate segment theorem, angle in a semicircle is a right angle, angles in the same segment are equal) are proved by introducing the centre. The point of knowing those theorems is to avoid having to do so yourself.

4. In this problem we are considering arrangements of points in a 3×4 rectangle (points do not necessarily have integer coordinates). An arrangement is *spacious* if all pairs of points are greater than $\sqrt{5}$ apart and is called *crowded* otherwise.

(a) Show that there is a spacious arrangement of five points.
(b) Show that every arrangement of seven points is crowded.
(c) Show that every arrangement of six points is crowded.

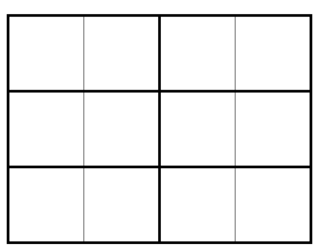
(a) **Solution:** Spreading the points out as far as possible seems sensible and indeed if we put a point at each of the four corners and one in the centre the distances between pairs of points are $3, 4, \frac{5}{2}$ and 5 which are all greater than $\sqrt{5}$ so the arrangement is spacious.

(b) **Discussion:** If we draw a few 3×4 rectangles and place the seven points in one-by-one (so that each new one is at least $\sqrt{5}$ from the others), we run out of space quite quickly which is a fairly convincing display that every arrangement of seven must be crowded. We might try and argue that we are adding new points in the 'best' way and since we run out of space in our examples it must be impossible to have a spacious arrangement of seven. While these might be quite convincing they are far from a proof: what do we mean by 'best' and how do we know we are adding the early points in such a good way.

This is a common trap with many combinatorial arguments: we may come up with an arrangement/process that seems 'best' (for example small changes may only make the situation worse) but this does not prove that there is not some totally different arrangement/process which is better.

Here we would like to show that there are two points which are close together. To do so we might split the rectangle up into a few smaller regions and hope to show that some of the points are in the same small region and so are not far apart. We want any two points inside the same small regions to be at most $\sqrt{5}$ apart: examples of such a region would be a circle with diameter $\sqrt{5}$ and a 1×2 rectangle (which in fact could be placed inside such a circle) since the diagonal of such a rectangle has length $\sqrt{1^2 + 2^2} = \sqrt{5}$.

Solution: We may split the 3×4 rectangle up into six 1×2 rectangles:



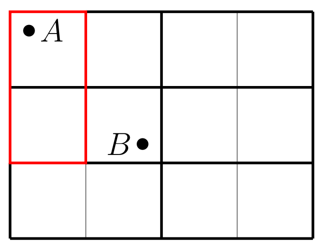
If we place seven points into the 3×4 rectangle then one of these 1×2 rectangles must contain at least two points. But any two points inside the same 1×2 rectangle are at most $\sqrt{5}$ apart.

Remarks: Here we used the fact that if we place seven points into six regions then some region must contain at least two points. This idea is commonly know as the *pigeonhole principle*: if we place $n + 1$ pigeons (in this case the points) into n pigeonholes (in this case the 1×2 rectangles) then some pigeonhole contains at least two pigeons.

We can take this further: if we place $kn+1$ pigeons (k is an integer) into n pigeonholes then there is a pigeonhole containing at least $k+1$ pigeons. In the set up of this question, if we place 19 points into the 3×4 rectangle then one of our six 1×2 rectangles contains at least four of the points (this shows that if we place 19 points into the 3×4 rectangle then there are four points forming a quadrilateral with area at most 2).

(c) **Discussion:** Although the previous argument does not work it does still give us some information: if we had a spacious arrangement of six points then each of the six 1×2 rectangles must contain exactly one of the points. This is true for any dissection of the rectangle into six 1×2 rectangles.

Consider the point A in the top left 1×2 rectangle: it can be in one of two 1×1 squares (either the left one or the right one). Suppose it is in the left one. Now consider the point B in the middle left 1×2 rectangle (which we think of as two 1×1 squares): B cannot be in the left 1×1 square otherwise it will be within $\sqrt{5}$ of point A (they would both be in the red 2×1 rectangle drawn below) so B must be in the right 1×1 square.

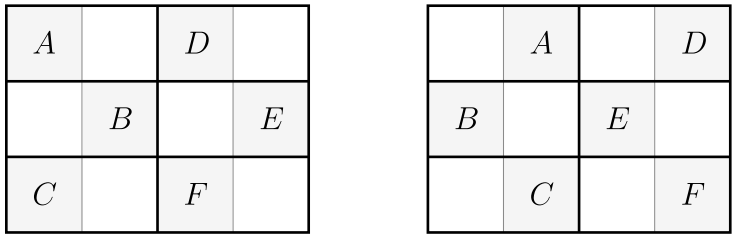


We can repeat this argument to nail down six 1×1 squares each of which contain exactly one of the six points. We can argue similarly if we suppose A is in the right 1×1 square.

Solution: Suppose, for contradiction, we have a spacious arrangement of six points. We split the 3×4 rectangle into six 1×2 rectangles as in the first diagram. In a spacious arrangement no two points can be in the same 1×2 rectangle so each 1×2 rectangle contains exactly 1 of the six points. Call the point in the top left 1×2 rectangle A, the point in the middle left 1×2 rectangle B, the point in the bottom left 1×2 rectangle C, the point in the top right 1×2 rectangle D, the point in the middle right 1×2 rectangle E and the point in the bottom right 1×2 rectangle F.

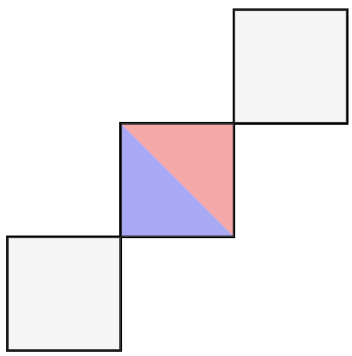
Each of the six 1×2 rectangles splits into two 1×1 squares: a left one and a right one. We will say A is 'on the left' if it is in the left 1×1 square of its 1×2 rectangle (the top left one) and 'on the right' otherwise. Similarly for the other points. We use throughout that no two points can be in the same 1×2 rectangle.

First suppose that A is on the left. Then B must be on the right and so C must be on the left. As B is on the right, E must be on the right and so D and F are on the left. If instead we had supposed that A was on the right then B must be on the left and so C must be on the right. As A is on the right, D must be on the right so E must be on the left and so F must be on the right. We thus have two possibilities (each shaded squares contains the point whose label is there):



In both cases there are three 1×1 shaded squares that form an upwards-right diagonal (C, B, D in the first case; C, E, D in the second). We show it is impossible to place three points inside such a diagonal without two being within $\sqrt{5}$ of each other.

Call the point in the bottom left square P, the one in the middle square be Q and the one in the top right square be R.



Every point in the blue triangle is within $\sqrt{5}$ of the whole of the bottom left square while every point in the red triangle is within $\sqrt{5}$ of the whole of the top right square. Thus either Q is within $\sqrt{5}$ of P or within $\sqrt{5}$ of R.

Remarks about the write-up: When writing-up a solution for a problem like this, which requires lots of explanation, it is easy to fall into the trap of producing large chunks of prose that are hard to understand: often called a ‘combinatorial essay’. There are various ways to make things easier for the reader. Labelling objects that are being discussed avoids ambiguous references (which can happen if a word such as ‘it’ is overused). Drawing diagrams to which you can refer helps with understanding. For example the checkerboard diagram containing A, B, C, D, E and F would signal to a marker that your argument was along the right lines. If there are key facts that are repeatedly used (such as no two points being in the same 1×2 rectangle) then it is worthwhile emphasising this to the reader.