United Kingdom
Mathematics Trust

# Intermediate Mathematical Olympiad

## Cayley paper



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## Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

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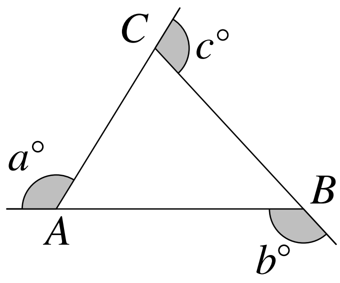
**1.** In the triangle $ABC$, the three exterior angles $a°$, $b°$ and $c°$ satisfy $a + b = 3c$.

Prove that the triangle $ABC$ is right-angled.

[Note: The diagram has been included to illustrate the labelling only and is not drawn to scale.]




Solution

The exterior and interior angles at each corner of the triangle add up to 180°, so the three interior angles have the values $180 - a$, $180 - b$ and $180 - c$.

The angles in any triangle add up to 180° so $(180 - a) + (180 - b) + (180 - c) = 180$.

Simplifying this equation gives $a + b + c = 360$.

The question tells us that $a + b = 3c$, so substituting for $a + b$ gives $3c + c = 360$, which simplifies to $4c = 360$.

Dividing both sides of this equation by 4 gives $c = 90$.

We know that the angle inside the triangle at $C$ is equal to $180 - c = 180 - 90 = 90$. Therefore triangle $ABC$ has a right angle at $C$.

**2.** The digits 1, 2, 3, 4, 5, $A$ and $B$ are all different and nonzero. Each of the two six-digit integers '$A$12345' and '12345$A$' is divisible by $B$.

Find all possible pairs of values of $A$ and $B$.

SOLUTION

Since the digits 1, 2, 3, 4, 5, $A$ and $B$ are all different and nonzero, $A$ and $B$ must be two of 6, 7, 8 and 9.

The integer '$A$12345' is odd and is divisible by $B$, so $B$ must be odd too. This means $B$ will either be 7 or 9.

Suppose $B = 7$. This means that '12345$A$' is divisible by 7, but the only multiples of 7 starting with 12345 are 123452 and 123459, which are $17636 \times 7$ and $17637 \times 7$, respectively. Since $A$ cannot be 2, $A$ must be 9.

We must now check that '$A$12345' is divisible by 7 and this is true since 912345 is $130335 \times 7$.

Suppose instead that $B = 9$. This means that '12345$A$' is divisible by 9, but the only multiple of 9 starting with 12345 is 123453, which is $13717 \times 9$, and $A$ cannot be 3, so $B$ cannot be 9.

Therefore there is only one solution and that is $A = 9$, $B = 7$.

**3.** Four friends rent a cottage for a total of £300 for the weekend. The first friend pays half of the sum of the amounts paid by the other three friends. The second friend pays one third of the sum of the amounts paid by the other three friends. The third friend pays one quarter of the sum of the amounts paid by the other three friends.

How much money does the fourth friend pay?

SOLUTION

Let $f_a$ represent the amount of money the $a^{\text{th}}$ friend pays, in pounds. Therefore the sum of the amounts paid by the other three friends is $300 - f_a$ pounds.

The first friend pays half of the amount paid by the other three friends so $f_1 = \frac{1}{2}(300 - f_1)$. Multiplying both sides by 2 gives $2f_1 = 300 - f_1$. Adding $f_1$ to both sides gives $3f_1 = 300$ and so $f_1 = 100$.

Similarly, the second friend pays one third of the amount paid by the other three friends so $f_2 = \frac{1}{3}(300 - f_2)$. Multiplying both sides by 3 gives $3f_2 = 300 - f_2$. Adding $f_2$ to both sides gives $4f_2 = 300$ and so $f_2 = 75$.

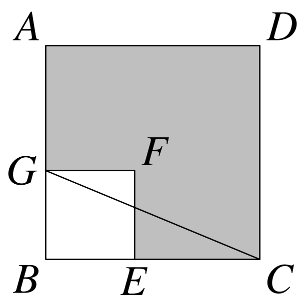
Finally, the third friend pays a quarter of the amount paid by the other three friends so $f_3 = \frac{1}{4}(300 - f_3)$. Multiplying both sides by 4 gives $4f_3 = 300 - f_3$. Adding $f_3$ to both sides gives $5f_3 = 300$ and so $f_3 = 60$.

This means that the fourth friend must pay $300 - 100 - 75 - 60$ pounds.

So the fourth friend pays £65.

**4.** Two squares $ABCD$ and $BEFG$ share the vertex $B$, with $E$ on the side $BC$ and $G$ on the side $AB$, as shown.
The length of $CG$ is 9 cm and the area of the shaded region is 47 $\text{cm}^2$.

Calculate the perimeter of the shaded region.




Solution

Let the length of $BC$ be $x$ cm.

Triangle $BCG$ is right-angled and $CG = 9$ cm, so applying Pythagoras' Theorem we have $BG^2 = 9^2 - x^2 = 81 - x^2$.

The area of the shaded region = the area of square $ABCD$ − the area of square $BEFG$.

Therefore $47 = x^2 - (81 - x^2)$.

This simplifies to $47 = x^2 - 81 + x^2$.

Adding 81 to both sides and simplifying gives $128 = 2x^2$.

So $x^2 = 64$.

Since $x$ represents a length, it cannot be negative, so $x = 8$.

Since $EF = BG$ and $FG = EB$, the perimeter of the shaded region is equal to the perimeter of square $ABCD$, which is $4 \times 8$.

Therefore the perimeter is 32 cm.

**5.** A ladybird is free to fly between the $1 \times 1$ cells of a $10 \times 10$ square grid. She may begin in any $1 \times 1$ cell of the grid. Every second she flies to a different $1 \times 1$ cell that she has not visited before.

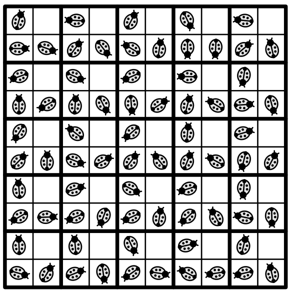
Find the smallest number of cells the ladybird must visit, including her starting cell, so that you can be certain that there is a $2 \times 2$ grid of adjacent cells, each of which she has visited.

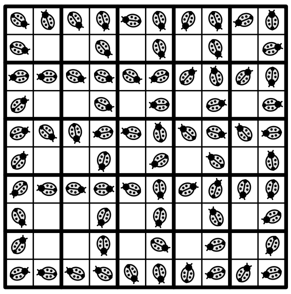
Solution

Suppose each $1 \times 1$ cell starts coloured white and when the ladybird visits any cell, including the starting cell, a ladybird symbol is marked in that cell.

We will show that is is possible to have an arrangement of 75 ladybird symbols without a completely filled $2 \times 2$ grid anywhere but that as soon as you have any arrangement of 76 ladybird symbols, there must be a completely filled $2 \times 2$ grid somewhere in the arrangement.

It is possible to find many different arrangements of 75 ladybird symbols such that no $2 \times 2$ grid contains 4 ladybird symbols. For example:





Since it is possible to use 75 ladybird symbols and not complete a $2 \times 2$ grid, the smallest number of cells visited that guarantees a completely filled $2 \times 2$ grid somewhere in the $10 \times 10$ grid must be greater than 75.

We now show that the smallest number to guarantee a complete $2 \times 2$ grid somewhere in the $10 \times 10$ grid is less than or equal to 76.

Split the $10 \times 10$ grid into 25 separate $2 \times 2$ grids.

The maximum number of symbols that these 25 separate $2 \times 2$ grids can hold without any of them being completely filled is $25 \times 3$, which is 75 ladybird symbols. So any attempt to insert 76 ladybird symbols into the $10 \times 10$ grid must completely fill one of these $2 \times 2$ grids.

Therefore, 76 is the smallest number of cells visited that guarantees a complete $2 \times 2$ grid somewhere.

**6.** Martha and Nadia play a game. Each has to make her own four-digit number, choosing her four digits from eight "digit cards" labelled 1-8. First Martha chooses her thousands digit, and then Nadia chooses her thousands digit. Next, Martha chooses her hundreds digit from the remaining six cards, and then Nadia chooses her hundreds digit. This process is repeated for the tens and finally the units digits of their numbers. The two four-digit numbers produced are then added together. Martha wins if the sum is not a multiple of 6; Nadia wins if the sum is a multiple of 6.

Determine which player has a winning strategy (that is to say, which player can guarantee that she will win no matter which digits the other player chooses).

Solution

For a number to be a multiple of 6, it must be a multiple of 2 and a multiple of 3.

Let Martha's four-digit number be '$abcd$' and Nadia's number be '$efgh$'.

We will show that the sum of these two four-digit numbers is actually always a multiple of 3.

The sum of the two four-digit numbers is $1000a + 100b + 10c + d + 1000e + 100f + 10g + h$.

This can be re-written as $999a + 99b + 9c + 999e + 99f + 9g + a + b + c + d + e + f + g + h$, which is $3(333a + 33b + 3c + 333e + 33f + 3g) + a + b + c + d + e + f + g + h$. This is a multiple of 3 added to $a + b + c + d + e + f + g + h$.

But we know that $a$, $b$, $c$, $d$, $e$, $f$, $g$ and $h$ are the digits 1, 2, 3, 4, 5, 6, 7 and 8 in some order. So $a + b + c + d + e + f + g + h = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$, which is also a multiple of 3.

This means that the sum of the two four-digit numbers is always a multiple of 3, whatever cards Martha and Nadia choose.

So, for Nadia to win the game, she needs to be able to ensure that the sum of the two four-digit numbers is a multiple of 2, and so is automatically a multiple of 6.

A winning strategy for Nadia is to always choose an even number if Martha just chose an even number and an odd number if Martha chose an odd number. This is always possible as we start with four even and four odd numbers in the digits from 1 to 8. This strategy will guarantee that the sum of the two four-digit numbers is even and so a multiple of 6.

N.B. There are many winning strategies for Nadia, some of which replace the need to prove the general property that two four-digit numbers, chosen in the way described, always add up to a multiple of 3, with a restricted argument concerning particular pairings of numbers that tell Nadia what to choose if Martha chooses one of the digits in the pair. For example $\{1, 5\}$, $\{2, 8\}$, $\{3, 7\}$ and $\{4, 6\}$.