Maclaurin Olympiad 2021 Markers Report

Comments from the Marking Team

Both candidates and their teachers will find it helpful to know something of the general principles involved in marking Olympiad-type papers. The preliminary paragraphs therefore serve as an exposition of the 'philosophy' which has guided both the setting and marking of all such papers at all age levels, both nationally and internationally.

What we are looking for, essentially, is solutions to problems. This approach is therefore rather different from what happens in public examinations such as GCSE and A level, where credit is given for the ability to carry out individual techniques regardless of how these techniques fit into a protracted argument. Such marking is cumulative; a candidate may gain 60% of the available marks without necessarily having a clue about how to solve the final problem. Indeed, the questions are generally structured in such a way as to facilitate this approach, divided into many parts and not requiring an overall strategy for tackling a multi-stage argument.

In distinction to this, Olympiad-style problems are marked by looking at each question synoptically and deciding whether the candidate has some sort of overall strategy or not. An answer which is essentially a solution, but might contain either errors of calculation, flaws in logic, omission of cases or technical faults, will be marked on a '10 minus' basis. One question we often ask is: if we were to have the benefit of a two-minute interview with this candidate, could they correct the error or fill the gap? On the other hand, an answer which shows no sign of being a genuine solution is marked on a '0-plus' basis; up to 3 marks might be awarded for particular cases or insights. It is therefore important that candidates taking these papers realise the importance of the rubric about trying to finish whole questions rather than attempting lots of disconnected parts.

Question 1

This question, which involves solving simultaneous quadratic equations, was answered confidently by many candidates. There are several methods which will give the correct solutions for x and y.

However, there were some common errors, such as cancelling y from $y^2 = -4xy$ to obtain $y = -4x$. This loses the two solution pairs when $y = 0$, and the way to avoid this mistake is to factorise the equation $y^2 + 4xy = 0$ as $y(y + 4x) = 0$, from which it is possible to deduce that either $y = 0$ or $y + 4x = 0$, each of which yields two solutions.

Another potential source of error is to divide by an expression. For example, it is possible to write the first equation as $2y = x - \frac{1}{x}$ and substitute this into the other equation to produce a quartic. There is a potential problem here since it might be possible for x to be zero. This cannot, in fact, happen, since this would contradict the first equation. However, it would certainly be unwise to divide by an expression such $x + 2y$ since there is no guarantee that this is non-zero.

A third source of error is to take the square root of an expression. For example, it is true that $(x-y)^2 = 1+y$, but if this is replaced by $x-y=\sqrt{1+y}$ you are assuming that $x-y$ is positive, and you will lose the solutions $(-1,0)$ and $\left(-\frac{1}{3},\frac{4}{3}\right)$.

There is a more subtle point. In any sequence of algebraic manipulations on a set of equations, what is really being shown is that, if these equations are satisfied, then the variables involved must take certain values. However, this does not mean that these values actually satisfy the equations.

For example, if you were to begin with $x+y=5, xy=4 \text{ and } 2x-y=7$, you could deduce, from the first two equations, that $(x,y)=(1,4) \text{ or } (4,1)$. It turns out only the second of these pairs satisfies the third equation. So, strictly speaking, you should always check answers with the original equations just to ensure that this sort of thing is not happening. The fact that you do not need to do this in ordinary school mathematics is because some unstated theorems about – say – simultaneous equations are being assumed. It is also quite a good idea to check your answers as a matter of course, just to make sure that you have not made an arithmetical slip. However, candidates were not penalised in the mark scheme for failing to check their answers.

Question 2

This is an enumeration problem, and, as ever, the requirement is to make sure that every case is counted *once* and *only once*. It is clear that you should consider two situations: lines from a vertex and lines which are not from a vertex. The former are relatively easy to count but the latter need a little more care. The question was done well, but candidates did not always take heed of the requirement on the title page to 'give full written solutions, including mathematical reasons why your method is correct'. Hence a script which simply announced that the number of choices is $3\times3\times3\times4\times2=216$ would be given very few marks.

There were some common mistakes. Several scripts gave the number of vertex links as $3+3+3$, presumably on the basis that there were three choices for each of three vertices, but interpreting the English word 'and' as meaning 'plus'.

For the non-vertex links, it was helpful to give the diagram showing the remaining six points, two on each side, which had to be joined, and then to consider the choices for each of two points on an arbitrary side. An explanation was necessary of why there are four choices for the first such point but only two for the second. Finally, it was essential to say why the last link was then determined. Again, there were some mistakes such as $4+2$ rather than 4×2, and there were other scripts which gave the final answer as $(3\times3\times3)+(4\times2\times1)$. As always, you are more likely to get marks for incorrect numerical answers if you explain carefully what the logic of your method is.

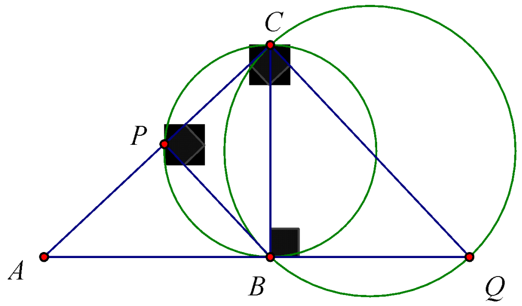
There were a number of candidates who thought this question was about combinations and then tried to reduce the very large number by dividing to avoid counting configurations twice. This is, perhaps, a matter of too much knowledge applied indiscriminately.

Question 3

This problem on geometry was unpopular, but the key result, the alternate segment theorem, is in the school curriculum, at least in England. In fact, you can avoid using this if you draw the radius of a circle, know that it is perpendicular to a tangent and then use angles in the same segment and isosceles triangles. It results in three equal angles and three similar triangles from which the result can be deduced. Competitors who were familiar with the requisite results did well in finishing this question, and for those who also knew the tangent-secant theorem this was a gift.

The real issue is that virtually no time is spent in school solving problems like this since they do not crop up apart from in competitions of this nature. Unfortunately, the only advice we can offer is to make use of the resources available from the UKMT and other sources in practising this skill.

A very large number of candidates began by drawing this diagram, which assumes that BC and CQ are diameters of their respective circles. Unfortunately, this is a very special case which made the rest of the argument trivial.



Question 4

This is an unusual question since it can be approached from two directions – the first of which might be described as *bottom-up* and the second as *top-down*.

The *bottom-up* argument realises that the first two terms in the sequence are squares which differ by 8 and then examines the table of squares to see how this can happen. It is not hard to convince yourself that the only pertinent example is 1 and 9 but a bit harder to establish it without any doubt. The key is the fact that the difference between consecutive squares increases, so you need only look at the first few terms in the sequence (and it turns out that the relevant squares are not consecutive). There is an alternative approach which solves the Diophantine equation $b^2 - a^2 = 8$ by factorising. Once you know that $k = 1$ you then have to show that all subsequent terms of the sequence are squares.

The *top-down* approach begins by trying to find the n^{th} term of the sequence in terms of k and examines how this can be square for all values of n. Here a key step is completing the square in the formula and then appealing to the fact that the differences between consecutive squares increase. This fixes the value of k.

This question was tackled with confidence by many candidates. In both approaches, it was necessary to find a closed-form expression for the sum of an arithmetic progression, and the markers were relaxed about allowing a variety of approaches here, and, in particular, the method of differences, when the second difference is shown to be constant. Marks were deducted for showing that $k = 1$ produced squares but failing to rule out other values of k, for being imprecise about consecutive and non-consecutive squares and for incomplete treatments of the identification of 1 and 9 as the only squares which differ by 8. Induction was not required, of course, but candidates who elected to use this method were penalised if they had not checked the base case.

Question 5

This was perceived as the hardest question on the paper and was attempted by the fewest candidates. However, there were many good attempts at solving it. Hardly anybody found the first solution, but it is included as it is insightful from a geometrical viewpoint. Various methods using scale factors led to solutions like the second, which used the similarity ratio to write the sides of the lawn as multiples of those of the playground. The most popular approach used triangles at vertices Q' and R' which were also similar to the playground and lawn. The third solution is a particular elegant form of this approach as it recognises that the quadrilaterals at the vertices are rhombi.

It was deemed appropriate to penalise scripts which did not explain why the playground and lawn triangles were similar. There are actually several steps here. Since the path has constant width, the sides of the playground are parallel to those of the lawn. Hence the angles of the lawn are equal to the angles of the playground. Hence the lawn triangle and the playground triangle are similar, and, finally, the sides are in the same proportion.

To why this is important, suppose that the shape of the playground was a quadrilateral. It would follow that the angles of the lawn were equal to those of the playground, but it would not follow that the sides were in proportion. Nor, for that matter, would there be a centre of similarity, as in the triangular case.

It was pleasing to see that very few scripts attempted to use Pythagoras on the lawn triangle in the second solution. This leads to some heavy algebra.

Question 6

This question was demanding since it divided into three distinguishable parts which needed different combinatorial techniques. It attracted more attempts than question 5, but it was more difficult since there is a lot of work to do, and it turns out to be useful to divide it into three independent parts.

In a question like this, there is a temptation to write essays. These are very difficult to read (and particularly online) and you should make an effort to break the narrative up into parts. Often it is useful to make a claim (such as ‘the cat and mouse will always have made the same number of moves’, or ‘the parity of a square for a particular animal is fixed’) and then prove it carefully.

It is helpful to supplement your argument using diagrams. These are very useful for showing, for example, why a particular route through the grid allows the cat and mouse to meet; but you need to explain why this exists in all appropriate cases, and not just for the particular diagram you have supplied.

Having analysed this problem, you should realise that there are essentially three things to deal with and then structure your argument accordingly.

The first part, worth 2 marks, is to show that the cat and mouse cannot meet if $m+n$ is odd. The best way to do this is to use a standard chessboard colouring. The two animals are constrained to squares of the same colour, but if $m+n$ is odd they begin on different colours so they cannot meet. Note that it is not enough to claim that the ‘best possible strategy’ (such as keeping close to the boundaries of the grid) does not work. It is possible that there is *another* method which does work. In other words, your argument should apply to any paths taken by either of the animals.

The second part, worth 4 marks, is to show that, if m and n are both even, then the animals can never meet. A useful idea here is the *parity* of a square (for each animal in turn) – is it reachable in an odd or an even number of moves from the starting point? Are there any squares which can be reached in an odd number of moves using one route and an even number using another? It turns out that this is not the case, but you need to explain why. Note that at any point the animals will have made the same number of moves. If it turns out that the parity of each square is fixed, but that for the cat and the mouse is opposite, then they cannot meet, and that is what happens when m and n are both even.

The third part is also worth 4 marks, and it shows that, if m and n are both odd, the cat and mouse can always meet. This turned out to be the hardest part of the question. Note that the parity argument does not lead to a contradiction, but that does not mean that the animals can meet. There were some attempts using co-ordinates which were nearly correct – for example, claiming that the difference between coordinates could be +2, 0 or -2 with complete freedom. This is a promising idea but the problem arises when the animal is at the edge of the grid; the first move, for example, can only result in a change of -2. It might be sensible to have the mouse toggling between two squares and then show how the cat can reach it.

Several candidates claimed that the cat and mouse could meet if, and only if, $m = n$, but this is wrong on both counts. Others claimed that when m and n are both odd, the cat and mouse can always meet on the central square. When $m = 3$ and $n = 5$, for example, neither animal can reach this square, but they can meet elsewhere.

This is a challenging question and you should feel pleased if you have solved any of the three parts.