United Kingdom
Mathematics Trust

Intermediate Mathematical Olympiad

Hamilton paper

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

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1. Naomi has a broken calculator. All it can do is either add one to the previous answer, or square the previous answer. (It performs the operations correctly.) Naomi starts with 2 on the screen. In how many ways can she obtain an answer of 1000?

SOLUTION

Solution 1

Let the notation $a \rightsquigarrow b$ mean "keep adding 1 to a until you reach b" (where $b \geq a$), and $a \hookrightarrow b$ mean "square a to get b".

For example, one route to 1000 is $2 \rightsquigarrow 5 \hookrightarrow 25 \rightsquigarrow 30 \hookrightarrow 900 \rightsquigarrow 1000$.

Note that $31^2 = 961 < 1000$ and $32^2 = 1024 > 1000$.

If **she does not square any numbers** then there is **one** way to obtain 1000: $2 \rightsquigarrow 1000$.

If **she squares a number once** then her route to 1000 looks like this:

$$2 \rightsquigarrow n \hookrightarrow n^2 \rightsquigarrow 1000.$$

It is clear that n can be any number with $n \geq 2$ and $n^2 \leq 1000$, so $n \leq 31$, that is, n can take any value from 2 to 31 inclusive.

There are therefore **30** ways she can obtain 1000 if she squares once.

If **she squares a number twice** then her route to 1000 looks like this:

$$2 \rightsquigarrow a \hookrightarrow a^2 \rightsquigarrow b \hookrightarrow b^2 \rightsquigarrow 1000,$$

with $2 \leq a < a^2 \leq b < b^2 \leq 1000$.

Since $b^2 \leq 1000$, $b \leq 31$, so $a^2 \leq 31$, that is $a \leq 5$.

If $a = 2$, we have $2 \hookrightarrow 4 \rightsquigarrow b \hookrightarrow b^2 \rightsquigarrow 1000$ so $4 \leq b \leq 31$ (so there are 28 choices for b).

If $a = 3$, we have $2 \rightsquigarrow 3 \hookrightarrow 9 \rightsquigarrow b \hookrightarrow b^2 \rightsquigarrow 1000$ so $9 \leq b \leq 31$ (23 choices for b).

If $a = 4$, we have $2 \rightsquigarrow 4 \hookrightarrow 16 \rightsquigarrow b \hookrightarrow b^2 \rightsquigarrow 1000$ so $16 \leq b \leq 31$ (16 choices for b).

If $a = 5$, we have $2 \rightsquigarrow 5 \hookrightarrow 25 \rightsquigarrow b \hookrightarrow b^2 \rightsquigarrow 1000$ so $25 \leq b \leq 31$ (7 choices for b).

So in total there are $28 + 23 + 16 + 7 =$ **74** ways she can obtain 1000 if she squares twice.

If **she squares a number three times** then her route looks like this:

$$2 \rightsquigarrow a \hookrightarrow a^2 \rightsquigarrow b \hookrightarrow b^2 \rightsquigarrow c \hookrightarrow c^2 \rightsquigarrow 1000,$$

with $2 \leq a < a^2 \leq b < b^2 \leq c < c^2 \leq 1000$.

By an identical argument to before, we can conclude that $c \leq 31$ and $b \leq 5$, which leads to $a \leq 2$ (that is $a = 2$).

So the route becomes

$$2 \hookrightarrow 4 \rightsquigarrow b \hookrightarrow b^2 \rightsquigarrow c \hookrightarrow c^2 \rightsquigarrow 1000,$$

and we know $b = 4$ or $b = 5$.

If $b = 4$ there are 16 choices for c (as there were for b above) and if $b = 5$ there are 7 choices for c (as above).

So in total there are $16 + 7 = \mathbf{23}$ ways she can obtain 1000 if she squares three times.

She cannot square a number four or more times, because if she does then the route which leads to the smallest total is $2 \looparrowright 4 \looparrowright 16 \looparrowright 256 \looparrowright 256^2$, which is bigger than 1000.

Hence the total number of ways she can reach 1000 is $1 + 30 + 74 + 23 = \mathbf{128}$.

Solution 2

Define $f(n)$ to be the number of ways Naomi can reach the number n. We wish to find $f(1000)$.

Note that, if n is not a square, then $f(n) = f(n-1)$ (since she can only reach n by reaching $n-1$ then adding one). If n is a square, then $f(n) = f(n-1) + f(\sqrt{n})$ (since she could now also reach n by reaching $\sqrt{n}$ then squaring).

Note that the largest square less than 1000 is $961 (= 31^2)$, since $32^2 = 1024 > 1000$. Hence $f(1000) = f(961)$.

Starting from $n = 2$, we have, for the first few n:

$$f(2) = 1, f(3) = 1, f(4) = 2, f(5) = 2, \ldots f(8) = 2, f(9) = 3.$$

It is immediately apparent that, as the value of n increases, $f(n)$ changes value only when n is square.

So we have $f(\text{everything from 9 up to 15}) = 3$, then $f(16) = 3 + f(4) = 3 + 2 = 5$.

For every square reached between 25 and 64 (inclusive) [of which there are 4], $f(n)$ increases by 2 (since $f(n) = 2$ for $5 \leq n \leq 8$).

Then, as the value of n increases:

for every square between 81 and 225 [7 squares], $f(n)$ increases by 3 (since $f(n) = 3$ for $9 \leq n \leq 15$);

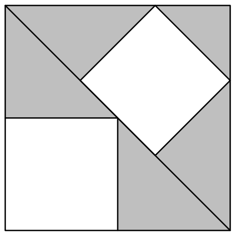
for every square between 256 and 576 [9 squares], $f(n)$ increases by 5 (since $f(n) = 5$ for $16 \leq n \leq 24$);

for every square between 625 and 961 [7 squares], $f(n)$ increases by 7 (since $f(n) = 7$ for $25 \leq n \leq 31$);

So $f(1000) = f(961) = f(16) + 2 \times 4 + 3 \times 7 + 5 \times 9 + 7 \times 7 = \mathbf{128}$.

2. The diagram shows two unshaded squares inside a larger square.

What fraction of the larger square is shaded?



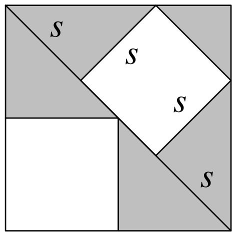
SOLUTION

Solution 1 Say the large square has side length $6x$, meaning it has area $36x^2$.

It is clear that the bottom-left white square has dimensions half those of the large square, i.e. the length of its sides is $3x$. This makes its area $(3x)^2 = 9x^2$.

By Pythagoras' Theorem, the main diagonal of the large square is $x\sqrt{72} = 6x\sqrt{2}$.

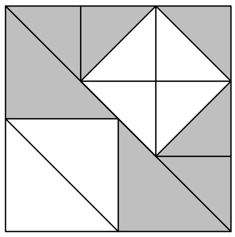
Say the top-right square has side length s. Then, because the triangles adjacent to it are isosceles (the two base angles are 45°), we know s is a third of the diagonal of the square, or $2x\sqrt{2}$. Hence the area of the upper white square is $8x^2$.



So the total white area is $17x^2$, which means that the total shaded area is $36x^2 - 17x^2 = 19x^2$. Hence the total shaded area is $\mathbf{\frac{19}{36}}$ of the square.

Solution 2

Split the bottom-left half of the square into four congruent triangles (we know they are congruent by the 'RHS' condition). Two of these cover the white square, so the white square covers half the area of the bottom-left triangle.



Split the top-right half of the large square into nine congruent triangles. (Again we know that all the triangles are congruent by the 'RHS' condition.)

The white square is covered by four of these triangles, so the white square comprises $\frac{4}{9}$ of the upper-right half of the large square and the shaded area comprises $\frac{5}{9}$ of the upper-right half of the square.

Hence the total shaded area is $\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{5}{9} = \frac{1}{4} + \frac{5}{18} = \frac{9}{36} + \frac{10}{36} = \mathbf{\frac{19}{36}}$ of the whole square.

3. For how many positive integers n less than 200 is n^n a cube and $(n+1)^{n+1}$ a square?

SOLUTION

For n^n to be a cube, we require either that n is a multiple of 3 or that n is a cube (or both).

For $(n+1)^{n+1}$ to be a square, we require either that $(n+1)$ is even or that $(n+1)$ is a square (or both).

We can take one from each of these pairs of criteria, giving four cases, and count the total:

(a) n is a multiple of 3 and $(n+1)$ is even:

This occurs exactly when n is an odd multiple of 3 (3, 9, 15, ...).

There are **33** such n less than 200.

(b) n is a multiple of 3 and $(n+1)$ is a square:

Checking all the squares between 1 and 200 as possibilities for $(n+1)$:

$n+1$ (square)	n	is n a multiple of 3?
4	3	yes
9	8	no
16	15	yes
25	24	yes
36	35	no
49	48	yes
64	63	yes
81	80	no
100	99	yes
121	120	yes
144	143	no
169	168	yes
196	195	yes

So there are nine of these.

However, we have counted five of them before ($n = 3, 15, 63, 99, 195$), so there are **4** new possible values of n.

(c) n is a cube and $(n+1)$ is even: for these we need n to be an odd cube:

$$n = 1, 27, 125$$
$$(n+1) = 2, 28, 126$$

There are three of these, but again we have counted one of them before ($n = 27$), so there are **2** possible new values of n.

(d) n is a cube and $(n+1)$ is a square:

It is easiest to check the possible values of $(n+1)$ for the cube values of n to see if any are square:

n (cube)	$n+1$	is $(n+1)$ a square?
1	2	no
8	9	yes
27	28	no
64	65	no
125	126	no

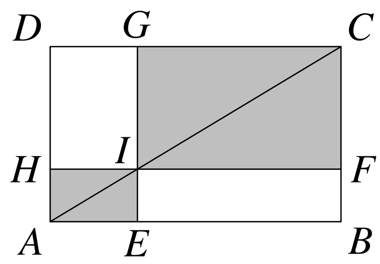
So there is **1** more value of n.

Hence in total there are $33 + 4 + 2 + 1 =$ **40** values of n such that n^n is a cube and $(n+1)^{n+1}$ is a square.

4. $ABCD$ is a rectangle with area 6 cm^2.

The point E lies on AB, F lies on BC, G lies on CD and H lies on DA. The point I lies on AC and is the point of intersection of EG and FH, and $AEIH$ and $IFCG$ are both rectangles. One possible diagram is shown to the right.

Given that the combined area of $AEIH$ and $IFCG$ is 4 cm^2, find all possible values for the area of rectangle $AEIH$ in cm^2.

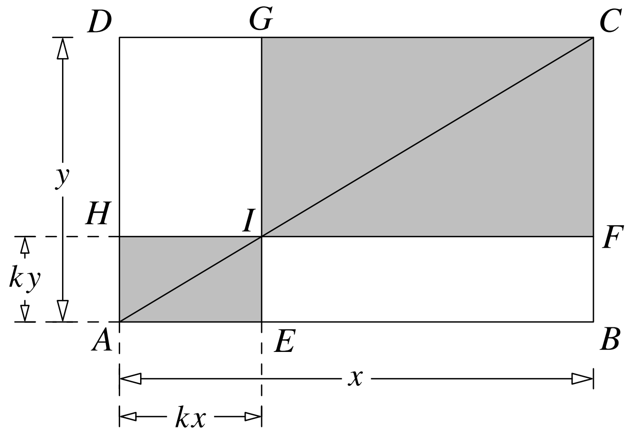


SOLUTION

Say that the lengths in cm of AB and AD are x and y respectively.

We are given that $xy = 6$.

Note that $AEIH$ is similar to $ABCD$, because they share a diagonal (meaning the ratios of their sides are the same). Then we know that, for some value of k (with $0 < k < 1$), $AE = kx$ and $AH = ky$.



Hence $EB = (1 - k)x$ and similarly $HD = (1 - k)y$.

We know that the shaded area is 4, and so

$$\begin{aligned}(ky)(ky) + (1 - k)x(1 - k)y &= 4\\ k^2xy + (1 - 2k + k^2)xy &= 4\end{aligned}$$

Since $xy = 6$, this becomes:

$$\begin{aligned}
6k^2 + 6(1 - 2k + k^2) &= 4\\
12k^2 - 12k + 2 &= 0\\
6k^2 - 6k + 1 &= 0\\
6\left(k - \frac{1}{2}\right)^2 - \frac{1}{2} &= 0\\
k - \frac{1}{2} &= \pm\sqrt{\frac{1}{12}}\\
k &= \frac{1}{2} \pm \frac{\sqrt{3}}{6}
\end{aligned}$$

(Since $0 < \sqrt{\frac{1}{12}} < \frac{1}{2}$, both of these values of k are between 0 and 1.)

The area of $AEIH$ is $k^2xy = 6k^2$.

If $k = \frac{1}{2} + \frac{\sqrt{3}}{6}$, the area $= 6k^2 = 6\left[\frac{1}{4} + \frac{\sqrt{3}}{6} + \frac{3}{36}\right] = \mathbf{2 + \sqrt{3}}$.

If $k = \frac{1}{2} - \frac{\sqrt{3}}{6}$, the area $= 6k^2 = 6\left[\frac{1}{4} - \frac{\sqrt{3}}{6} + \frac{3}{36}\right] = \mathbf{2 - \sqrt{3}}$.

It is straightforward to show that each of these areas is possible (e.g. by drawing a 6×1 rectangle).

Note: the same conclusion can be reached by equating the sum of the unshaded regions to 2 (they are, in fact, each equal to 1).

5. Find all real numbers x, y, z such that $x^2 + y^2 + z^2 = x - z = 2$.

SOLUTION

First, we claim that $y = 0$.

Proof: if $y \neq 0$ then $y^2 > 0$, so we have $x^2 + z^2 = 2 - y^2 < 2$.

Since $x = (2 + z)$, we have:

$$\begin{aligned}(2+z)^2 + z^2 &< 2\\ 4 + 4z + 2z^2 &< 2\\ z^2 + 2z + 1 &< 0\\ (z+1)^2 &< 0\end{aligned}$$

Which is not possible, so $\boldsymbol{y = 0}$.

Given that $y = 0$, we have $x^2 + z^2 = 2$. We can proceed in a similar way to above, substituting $x = (2 + z)$:

$$\begin{aligned}(2+z)^2 + z^2 &= 2\\ 4 + 4z + 2z^2 &= 2\\ z^2 + 2z + 1 &= 0\\ (z+1)^2 &= 0\end{aligned}$$

Hence $(z + 1) = 0$ so $\boldsymbol{z = -1}$, and from this we deduce $\boldsymbol{x = 1}$.

It is straightforward to check that these values satisfy the original equations, so the only solution is $(x, y, z) = \mathbf{(1, 0, -1)}$.

> **6.** Humpty buys a box of 15 eggs, with 3 rows and 5 columns. Each meal he removes one egg to cook and eat. If necessary, he moves one or more eggs in the box so that between meals there are always two lines of reflective symmetry. What is the smallest total number of extra egg moves he can make while he empties the box?
>
> **Note:** You must carefully justify that your answer is minimal; that it is impossible to make fewer extra egg moves while emptying the box.

Solution

Label the 15 egg spaces as in the diagram on the right.

A	B	C	D	E
F	G	H	I	J
K	L	M	N	O

First, note that, if there is currently an odd number of eggs in the box, there must be one in space H, since this is the only space which does not reflect onto (at least) one other space.

Also note that, every time Humpty removes an egg, the number of eggs in the box changes from even to odd (or vice versa).

So if the box has an even number of eggs in it, Humpty must, after removing an egg, move an egg into the central space to retain symmetry.

Humpty removes an egg from the box when it contains an even number of eggs seven times, so there must be at least seven egg moves.

Now consider eggs in the four corner spaces (A, E, K and O). If one of these spaces contains an egg, all four must (since they reflect onto each other), and if one of them is empty, all four must be.

So when Humpty first removes an egg from one of these spaces and does not move an egg into the space he has just created (which must occur at some point because he is emptying the box), he must move the eggs out of the remaining three spaces, which requires three egg moves (of course, one of these moves may be moving an egg into space H).

An identical argument can be applied to the four spaces B, D, L and N.

So we know that there must be at least 11 egg moves: three out of A, E, K and O, three out of B, D, L and N and at least five more into H (assuming two of the seven moves required into H were done when moving the sets of three eggs).

It turns out that it is possible to empty the box while making only 11 egg moves (for example see below), so we know **11 moves** must be the minimum.

step	remove from	move(s)	total moves
1	H	—	
2	G	I to H	1
3	H	—	1
4	F	J to H	2
5	H	—	2
6	A	E to H, K to G, O to 1	5
7	H	—	5

step	remove from	move(s)	total moves
8	G	I to H	6
9	H	—	6
10	B	D to H, L to G, N to I	9
11	H	—	9
12	G	I to H	10
13	H	—	10
14	C	M to H	11
15	H	—	11

A graphical representation of this is shown below.

● an egg ○ an egg which has just been moved elsewhere

□ an empty space ⊗ an egg which has just been removed from the box