United Kingdom Mathematics Trust

Cayley Olympiad 2021 Markers Report

Comments from the Marking Team

This year's paper was found to be more challenging than those in recent years. This is something that will please the setting committee, who had the intention of increasing the difficulty of the paper in the hope of spreading out the distribution of marks and lowering the average score. Although the paper was more demanding overall there was still ample opportunity for candidates of every level to demonstrate what they could do. Every candidate who took part should feel proud for taking on the challenge. There were excellent written solutions submitted to every question, demonstrating both ingenuity and rigour. It was clear that the final three questions provided a high level of challenge for even the strongest candidates.

Unfortunately, it remains the case that some candidates are reluctant to go beyond writing down calculations. It is vital when attempting an Olympiad problem that the solution presented is fully justified and clearly explained. In most cases this will involve writing down full sentences and on occasion providing detailed diagrams. At the very least it would be prudent for candidates to ask themselves the following question when checking a solution 'If someone else read my solution, could they reconstruct the question?'. If the answer is no then the solution is certainly lacking. Our advice for candidates is simple, do not hold back from writing down your thoughts and remember that doing mathematics does not necessarily mean solely doing calculations.

Question 1

This question was found to be accessible by the vast majority of students, however there were a higher number of misinterpretations than expected. It is vital that candidates read a question more than once and take note of key words or phrases before embarking on a solution. The two most common misinterpretations were thinking that the final four-digit numbers asked for must be prime themselves or that they must not contain a repeated digit.

There were several different strategies employed in this question and it is interesting to note that those candidates deviating from the official solution and taking a digit-by-digit approach to finding the relevant four-digit numbers had more work to do. There is an important lesson to be learnt here, when attempting to solve a problem it may not be sensible for candidates to dive in with their first approach.

There was a very fine line between achieving full marks and relatively few in this question. Some candidates omitted key steps in their argument, which meant penalties had to be applied even though the correct final answers had been obtained. It was common for candidates to forget to write down a relevant list of two-digit primes or to neglect to explain how to go from such a list to the final four-digit answers. For those candidates who did not state a list of two-digit primes, a common oversight was the failure to explain why two consecutive digits in the final four-digit numbers could not be the same (where a mention of divisibility by 11 would have sufficed). Even if a candidate believes a step in their argument to be obvious it is important that explanation is given, a marker cannot fill in the gap for them.

A small number of candidates thought that 1 was a prime a number and so these candidates should be reminded that a prime number has exactly two distinct positive factors, unlike 1 which has a single positive factor.

Question 2

This was an unusual question at Cayley level: tiling questions usually appear at a more senior level and it is rare for a such a question to consist only of a tiling that is possible. This is likely to be the most demanding second question set on a Cayley paper. Having said that, there were some excellent solutions and many candidates did achieve full marks. The most successful candidates were usually those who provided clear and detailed diagrams. Although it was possible to describe a correct tiling in words, candidates who chose not to draw any diagrams frequently lost marks due to unclear or imprecise explanations.

It was apparent that some candidates did not engage fully with the question and rushed a response along the lines of '2021 is not divisible by 3 or 4, so such a tiling is impossible'. If those candidates had spent more time thinking about the question, then we suspect they would have made progress.

It was common for candidates to argue that the numerical area of the larger rectangle is divisible by 6 and so because one triangular tile has area 6 cm^2 it is possible to tile the larger rectangle. This is a sensible idea and if the numerical area of the larger rectangle were not divisible by 6, it would have shown that Jack's task was impossible. Unfortunately, this argument on its own is not enough given that Jack's task is possible. Consider a 2cm by 6cm rectangle, this has a numerical area divisible by 6 but it is clearly impossible to form such a rectangle using the triangular tiles. It is important that in a problem like this, if a candidate believes the task is possible then they must give a detailed example showing that this is the case.

Question 3

This question yielded the most full mark responses of any question on the paper. In fact, there could have been many more candidates receiving full marks if they had presented their solutions with geometrical justifications. It was far too common for candidates to annotate a diagram and write down several equalities with no reference to the geometrical reasoning that yielded them. Candidates who gave no geometrical reasoning whatsoever were capped at 3 marks, even when they had correct equations leading to the correct final answer.

Some candidates were deducted marks for providing partial reasoning, most often this was because they claimed equality of two angles due to the presence of a pair of parallel lines. We expect more detail and specific geometrical terminology should be used, for example 'alternate' or 'corresponding' may be appropriate.

Although there were many different successful routes through this question, each of them hinged on four key steps:

1. **Stating that $\angle AEO$, $\angle BFE$ or $\angle OCF$ is equal to $\angle BCD = 4x°$ (various reasons)**
2. **Stating that $\angle OAE = \angle BCD$ due to the isosceles ΔOAE**
3. **Stating that $\angle OCB = \angle OAE$ as they are corresponding angles**
4. **Forming a linear equation by considering a subset of angles in ΔOCB, ΔOAE, ΔOCF or ΔOFB**

Where candidates were unable to find the correct answer, it was almost always because the second step was missing and they hadn't thought about the radius of the quarter circle. It is important for candidates to consider the starting configuration carefully and to think about how they utilise the fact that a quarter circle is involved. If a circle, or part of a circle, is included in a geometrical problem then the radius is likely to play an important role.

We saw many solutions which omitted a diagram completely and often such solutions become confused, and angles become incorrectly labelled. We would strongly recommend that all candidates draw a clear diagram at the start of any solution to a geometry problem.

Question 4

At this stage of the paper the level of demand began to rise and as expected we saw an increase in the number of non-attempts. This is a rather technical question for a Cayley paper and it was pleasing to see that more than forty candidates achieved full marks. There were more candidates close to achieving full marks but they omitted a key step in the argument: closed form expressions for $S(n)$ (for odd and even cases) were quoted without proof. We accepted a minimal level of proof, such as pairing up consecutive terms using brackets, but where reasoning was completely absent marks were deducted. It is worth repeating that even if a candidate believes a step in their argument is obvious, it must be written down for the marker to see.

This question really allowed candidates to shine mathematically: to obtain a full solution they needed to first prove closed form expressions for $S(n)$ and then then produce a three-part parity argument to find an infinite family of solutions. It was rather impressive that some candidates in this age group managed the task.

For less successful candidates it was clear that a lack of experience with equations having infinitely many solutions was a stumbling block. It was common for candidates to find a single pair of values $(a,\ b)$ that solved the equation but no more. On reflection, perhaps it would have been sensible to adjust the wording of the question to at least indicate the possibility of infinitely many solutions.

Question 5

This question aimed to really test the algebraic ability of the candidates and it certainly split the field. Pleasingly there were close to seventy full-mark solutions, and it proved to be slightly less demanding than question 4.

This was very much an all or nothing question, and the determining factor was whether a candidate could manipulate algebraic fractions correctly. Unfortunately, many candidates were unable to clear the denominators in the two given equations correctly. Once this step had gone wrong it was very difficult to award any marks at all. Some candidates attempted to work with fractions throughout their solution and at various stages in their proof incorrectly equated numerators or denominators. Due to the algebraic nature of the fractions involved, they had overlooked the possibility that the two fractions equated were equivalent but had distinct numerators and distinct denominators. Such an argument could score relatively few marks. It was excellent to see some candidates consider this possibility and go on to provide a full proof. Performing arithmetic with numerical fractions is something we would expect every candidate to be familiar with and have little trouble with, so being able to extend this to algebraic fractions is a reasonable expectation.

Only a small number of candidates seemed not to know the meaning of 'real numbers' and instead assumed that the variables were integer-valued. This was only penalised if their argument relied on that assumption. A penalty was applied if division by a quantity that could have been zero occurred (such as x), although there was no penalty for those who neglected to state that $p,\ q,\ y$ and z were non-zero. Some candidates may feel they provided an adequate proof but received relatively few marks, this was due to the incorrect conclusion that both xyz and pqr were equal to zero and thus their sum must be equal to zero too. To see that this is false take $x = 1,\ y = 2,\ z = 1,\ p = 2,\ q = 1$ and $r = -1$, this set of values satisfies the two equations given and $xyz + pqr = 0$.

Question 6

This proved to be a fitting final question for the paper, with just ten candidates achieving full-marks but a good number achieving partial marks. Although this is certainly among the most demanding combinatorics questions set on a Cayley paper, it only required an understanding of product rule for counting and complementary counting. If these are terms candidates are unfamiliar with, we recommend the UKMT Handbook 'Introduction to Combinatorics'.

It was clear that the high demand of this question was in part due to the variable number of sets of traffic lights. This meant that a brute force approach to the question was ruled out and conjectures from pattern spotting were rare. The general nature of the problem led to many non-attempts. In a problem as general as this it is often a good idea for candidates to specialise and consider what happens for small values of the variable. Although on its own not worth any marks, considering small values of n could have helped reveal the structure of sequences which alternate in colour and led to the required generalisation. It was rare we saw such an approach.

The question split nicely into three parts of differing difficulty: finding the total number of sequences which alternate in colour, finding the number of sequences which alternate in colour but contain zero red lights and finding the number of sequences which alternate in colour but contain exactly one red light. These three expressions could then be combined via complementary counting to find the final expression. When counting the total number of sequences which alternate in colour some candidates argued that there were three choices for the first colour, two for each after the first and so in total there were $3 + 2 + 2 + ... + 2$ sequences rather than the correct $3 \times 2 \times 2 \times \cdots \times 2$ sequences. It is important that these candidates reflect on why we want the product rather than the sum. A good number of candidates realised that there were only two sequences which alternated in colour but contained no red lights. Unfortunately, relatively few could then see how to apply this to the case of exactly one red light. A particular issue that arose in this case was that a red light at the beginning or end of the sequence must be treated differently to a single red light elsewhere in the sequence. This oversight was penalised, but candidates could still achieve a good score for the question.

There were a small number of candidates who provided nothing but calculations, and whilst the calculations were correct and led to the correct final expression, the lack of explanation meant full marks could not be awarded.