United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 1 : Friday, 2 December 2016

Time allowed $3\frac{1}{2}$ hours.

Instructions

- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.*
- *Each question carries 10 marks. However, earlier questions tend to be easier. In general you are advised to concentrate on these problems first.*
- *The use of rulers, set squares and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by your solutions in question number order.*
- *Staple all the pages neatly together in the top* left *hand corner.*
- *To accommodate candidates sitting in other time zones, please do not discuss the paper on the internet until 8am GMT on Saturday 3 December when the solutions video will be released at* `https://bmos.ukmt.org.uk`

Do not turn over until **told to do so.**

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2016/17 British Mathematical Olympiad

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1. The integers $1, 2, 3, \ldots, 2016$ are written down in base 10, each appearing exactly once. Each of the digits from 0 to 9 appears many times in the list. How many of the digits in the list are odd? *For example, 8 odd digits appear in the list* $1, 2, 3, \ldots, 11$.

2. For each positive real number x, we define $\{x\}$ to be the greater of x and $1/x$, with $\{1\} = 1$. Find, with proof, all positive real numbers y such that
$$5y\{8y\}\{25y\} = 1.$$

3. Determine all pairs (m, n) of positive integers which satisfy the equation $n^2 - 6n = m^2 + m - 10$.

4. Naomi and Tom play a game, with Naomi going first. They take it in turns to pick an integer from 1 to 100, each time selecting an integer which no-one has chosen before. A player loses the game if, after their turn, the sum of all the integers chosen since the start of the game (by both of them) cannot be written as the difference of two square numbers. Determine if one of the players has a winning strategy, and if so, which.

5. Let ABC be a triangle with $\angle A < \angle B < 90°$ and let Γ be the circle through A, B and C. The tangents to Γ at A and C meet at P. The line segments AB and PC produced meet at Q. It is given that
$$[ACP] = [ABC] = [BQC].$$
Prove that $\angle BCA = 90°$. *Here* $[XYZ]$ *denotes the area of triangle* XYZ.

6. Consecutive positive integers $m, m+1, m+2$ and $m+3$ are divisible by consecutive odd positive integers $n, n+2, n+4$ and $n+6$ respectively. Determine the smallest possible m in terms of n.