

UK SENIOR MATHEMATICAL CHALLENGE

Tuesday 8 November 2016

Organised by the United Kingdom Mathematics Trust
from the School of Mathematics, University of Leeds

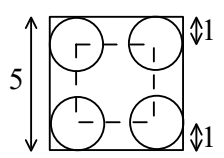
SOLUTIONS LEAFLET

This solutions leaflet for the SMC is sent in the hope that it might provide all concerned with some alternative solutions to the ones they have obtained. It is not intended to be definitive. The organisers would be very pleased to receive alternatives created by candidates.

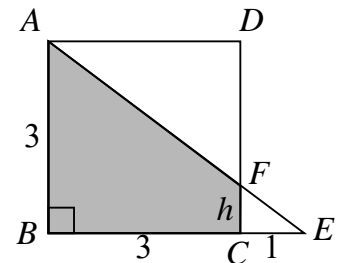
For reasons of space, these solutions are necessarily brief. There are more in-depth, extended solutions available on the UKMT website, which include some exercises for further investigation:

<http://www.ukmt.org.uk/>

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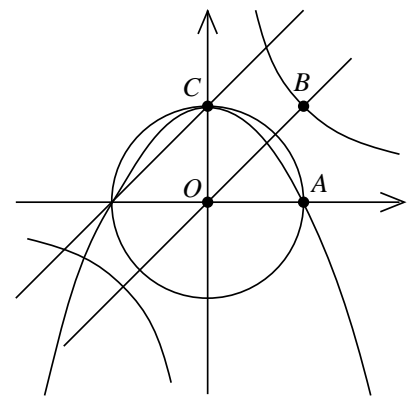
- B** Since the answer to 987654321×9 is 8 888 888 889, the digit 9 appears once.
- D** James bought a book for £5.50, so the normal price would be £5. On Friday, another copy of this book would therefore cost £4.50.
- B** As the circle rolls, the centre of the circle moves along four straight lines shown as dashed lines. Each dashed line has length $5 - (1 + 1)$ so the total distance travelled is 4×3 which is 12.
- C** One angle in Alex's triangle is 80° . Let α° be the smaller of the other two angles so $(100 - \alpha)^\circ$ is the third angle. The difference between these angles is then $(100 - 2\alpha)^\circ$. Considering each option:
A: $100 - 2\alpha = 0$ gives both α and $100 - \alpha$ to be 50. This triangle is therefore isosceles and not scalene.
B: $100 - 2\alpha = 60$ gives α to be 20 and $100 - \alpha$ to be 80. This is again isosceles.
Option D gives angles of 80, 0 and 100. Option E gives angles of 80, -10 and 110. Neither of these cases forms a triangle.
C: $100 - 2\alpha = 80$ gives α to be 10 and $100 - \alpha$ to be 90. All three angles are different so this is the correct option.
- A** From the available digits the squares could be 256, 324 or 625. Since the middle digits must be the same, the centre digit must be 2.

6. **D** Let F be the point of intersection of the lines AE and CD . Let the length of CF be h . Then, using similar triangles, $\frac{CF}{CE} = \frac{BA}{BE}$, so $\frac{h}{1} = \frac{3}{4}$ giving $h = \frac{3}{4}$. The shaded region $ABCF$ is a trapezium, so has area $\frac{1}{2} \left(3 + \frac{3}{4} \right) \times 3 = \frac{45}{8}$ which is $5\frac{5}{8}$.



7. **A** The number 2016^0 has value 1. As $2016 > 1$, $2016^{1/2} < 2016^1$. The values of their reciprocals, $2016^{-1/2}$ and 2016^{-1} are then in the opposite order. So the five options given are in numerical order, with 2016^{-1} , or $\frac{1}{2016}$, being the smallest.
8. **B** One way to count the number of small squares formed is to divide the large square into four quarters along its two diagonals. The number of small squares formed is $4 \times T_{n-1}$, where T_{n-1} is the $(n - 1)$ th triangular number. When $n = 7$, this is $4 \times \frac{1}{2}(6 \times 7)$ which is 4×21 . So 84 squares are formed.

9. **C** Let $O = (0,0)$, $A = (1,0)$, $B = (1,1)$, $C = (0,1)$ be the vertices of the square. The equation $x^2 + y^2 = 1$ gives a circle passing through A and C . The equation $y = x + 1$ gives a straight line passing only through C . The equation $y = -x^2 + 1$ gives a parabola passing through A and C . The equation $y = x$ gives a straight line passing through O and B . The equation $y = \frac{1}{x}$ gives a rectangular hyperbola which has two branches and passes only through B . So, only $x^2 + y^2 = 1$, $y = x$ and $y = -x^2 + 1$ have graphs passing through exactly two of the vertices of the square.



10. **B** None of the products for the first two rows and first two columns contains a factor of 5, so the bottom right cell must contain the 5. The prime factorisation of 112 is $2^4 \times 7$ and, as 7 is not a factor of 216 or 12, then 7 must be in the right cell of the middle row. The remaining 2^4 must be the product of two different numbers, namely 8 and 2. The 2 must be in the centre cell as 8 is not a factor of 12. The grid is now as shown above. The prime factorisation of 216 is $2^3 \times 3^3$ and the 3^3 must be the product of a 3 and a 9. The 3 must be in the top left cell as the product of the top row is 12 which is not a multiple of 9. Thus, the product of the three shaded cells is $3 \times 2 \times 5$ which is 30. The completed grid is as shown on the right.

8	2	7
		5

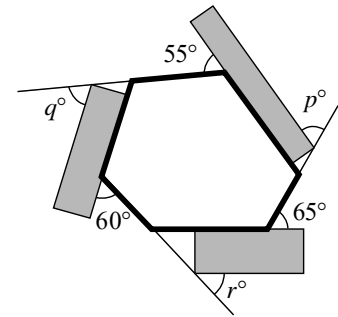
3	1	4	12
8	2	7	112
9	6	5	
216	12		

11. **E** For each square to be filled with the mean of the numbers in the adjacent squares, the differences between all five pairs of adjacent numbers must be equal. This common difference is $\left(\frac{25 - 10}{5} \right)$ which is 3. The grid is then
- | | | | | | |
|----|----|----|----|----|----|
| 10 | 13 | 16 | 19 | 22 | 25 |
|----|----|----|----|----|----|
- and the 19 is in the desired square.

12. **C** The prime factorisation of 2016 is $2^5 \times 3^2 \times 7$. To create the smallest square which is a multiple of 2016, the powers of each prime must be as small as possible and even, whilst also being at least as big as those in the prime factorisation of 2016. This gives $2^6 \times 3^2 \times 7^2$ which is $(2^3 \times 3 \times 7)^2$ or 168^2 .
13. **A** A quarter circle of radius r has length $\frac{2\pi r}{4}$ which is $\frac{\pi r}{2}$. The total length of the curve shown is then $\frac{\pi}{2}(1 + 1 + 2 + 3 + 5)$ which is 6π .

14. **D** The five options give the values of n to be considered. In option B, 11 is prime so that can be discounted. The options A, C and E are 6, 27 and 51 which are not prime and subtracting 2 from each of these gives 4, 25 and 49 which are also not prime. However in D, $n = 33$ which is not prime but $n - 2 = 31$ is prime.

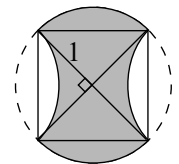
15. **B** A non-regular hexagon can be drawn on the diagram as shown. Three of the exterior angles of the hexagon are then 55° , 60° and 65° . Since corresponding angles on parallel lines are equal, the other three exterior angles are p° , q° and r° . The total of the exterior angles of any polygon is 360° . Hence
 $p + q + r + 55 + 60 + 65 = 360$ and so $p + q + r = 180$.



16. **D** The expression $\sqrt{2016} + \sqrt{56}$ can be written as $\sqrt{2^5 \times 3^2 \times 7} + \sqrt{2^3 \times 7}$ which is $\sqrt{4^2 \times 3^2 \times 2 \times 7} + \sqrt{2^2 \times 2 \times 7}$. This simplifies to $12\sqrt{14} + 2\sqrt{14}$ which is $14\sqrt{14}$ and, using index notation, this can be written as $14^{3/2}$. Hence $k = \frac{3}{2}$.

17. **D** One way to count the possible codes is in descending numerical order of the three-digit codes. The list begins: 789; 689, 679, 678; 589, 579, 578, 569, 568, 567; Each initial digit n produces part of the list with the $(8 - n)$ th triangular number of possible codes, where $n \leq 7$. The total number of possible codes is then the sum of these triangular numbers
 $1 + 3 + 6 + 10 + 15 + 21 + 28$ including 1 code starting with the digit 7, all the way to 28 codes starting with the digit 1. The total number of codes that Aaron can choose is 84.

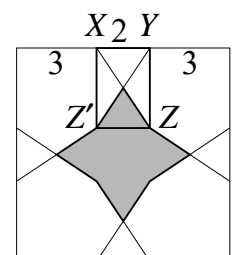
18. **E** The four arcs are of equal length and their end-points lie on a circle, so the four end-points can be joined to make a square. As two of the arcs are 'turned over', the two unshaded regions inside the square have areas equal to the two shaded regions outside the square.



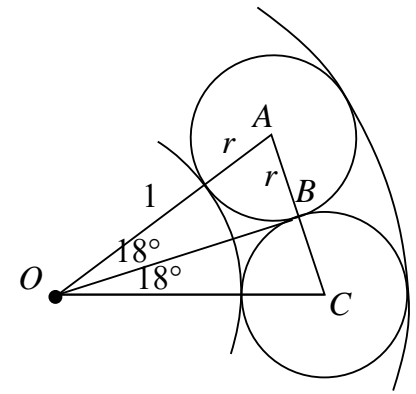
The total shaded area is therefore equal to the area of the square. The radius of the circle is given as 1 so, by Pythagoras' Theorem, the side-length of the square is $\sqrt{1^2 + 1^2} = \sqrt{2}$. So the area of the shaded region is $\sqrt{2} \times \sqrt{2} = 2$.

19. **D** Let S consist of h, j, k, l, m in ascending order of size. We want m to be as small as possible. Given three side-lengths, there is a quadrilateral with non-zero area with a specified fourth side-length if and only if the fourth side-length is less than the sum of the other three side-lengths. To ensure that j, k, l, m are not the side-lengths of such a quadrilateral, we must have $m \geq j + k + l$. Likewise, considering h, j, k, l , we must have $l \geq h + j + k$. Since the smallest possible values of h, j and k are 1, 2 and 3 respectively then $l \geq 1 + 2 + 3$ so 6 is the smallest value of l . Also $m \geq 2 + 3 + 6$ so 11 is the smallest value of m .

20. **E** Let the point Z' be directly below X , so that $XYZZ'$ is a rectangle. As the length of XY is 2 cm, the distance from Y to the nearest corner of the square is 3 cm. The area of $XYZZ'$ is $2 \text{ cm} \times 3 \text{ cm}$ which is 6 cm^2 . The diagonals XZ and YZ' split $XYZZ'$ into quarters and each has area $1\frac{1}{2} \text{ cm}^2$. The central grey octagon is formed from a square with side $Z'Z$ of length 2 cm together with four triangles, each of area $1\frac{1}{2} \text{ cm}^2$. The total area of the shaded octagon is $2 \times 2 + 4 \times 1\frac{1}{2}$ which is 10 cm^2 .



21. C As there are 10 discs, the adjacent lines drawn from the centre of the inner circle to the centre of each disc are separated by an angle of 36° . The line OB is a tangent to both the disc with centre A and the disc with centre C . So the points A, B and C lie on a straight line as angles OBA and OBC are both 90° .

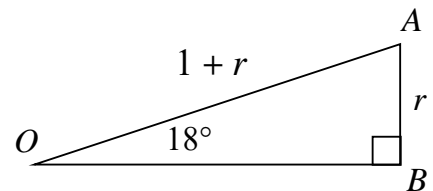


In the second diagram, from triangle OAB we have $\sin 18^\circ = \frac{r}{1+r}$ which rearranges to

$$\frac{\sin 18^\circ}{1 - \sin 18^\circ} = r.$$

The radius of the outer circle is

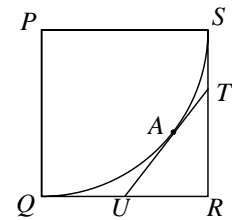
$$1 + 2r = 1 + \frac{2 \sin 18^\circ}{1 - \sin 18^\circ} = \frac{1 + \sin 18^\circ}{1 - \sin 18^\circ}.$$



22. C Suppose first that Cam is telling the truth, so Ben and Dan are both lying. Then Ben's statement is actually correct, as is Dan's. There is a clear contradiction. So we know that Cam is in fact lying. Therefore at least one of Ben and Dan is telling the truth. If Ben is telling the truth, then we learn that Dan is telling the truth. If Dan is telling the truth then we learn that Ben is telling the truth. So both Ben and Dan are telling the truth. This means that only Cam is lying.

23. E For the cuboid to be contained within a sphere of smallest possible radius, all eight vertices of the cuboid must lie on the sphere. The radius r of the smallest sphere is then half of the length of the body diagonal of the cuboid, so $r = \sqrt{1^2 + 5^2 + 11^2} = \sqrt{147}$. If the largest cube which will fit inside this sphere has side-length $2x$, then $r = \sqrt{x^2 + x^2 + x^2}$. Thus $3x^2 = 147$, so $x^2 = 49$ and so $x = 7$. The side-length of the largest cube is 14.

24. B Let the square have side-length 2, $RT = h$ and let A be the point of contact between TU and the circle. Two tangents to a circle which meet at a point are of equal length. So as $QU = 1$ so does AU . Similarly $TA = TS = 2 - h$. Applying Pythagoras' Theorem to triangle URT gives $1^2 + h^2 = (1 + 2 - h)^2$ so $1 + h^2 = 9 - 6h + h^2$ and therefore $8 - 6h = 0$ which gives $h = \frac{4}{3}$. The required ratio is then $4 : 3$.



25. D For n to be the smallest integer for which $7n$ has 2016 digits, $7n$ must start with 1, be followed by 2014 zeros and end with a digit a . When this number is divided by 7, the answer is formed from the repeating sequence of 6 digits 142857. The remainders also form a repeating sequence 3, 2, 6, 4, 5, 1. These sequences are repeated 335 times as 6×335 is 2010. The last 4 zeros (to make 2014 zeros in total) and the final a create the last section of the division as shown:

$$\begin{array}{r} \dots\dots\dots 1\ 4\ 2\ 8 \\ \dots\dots\dots 10^3 0^2 0^6 0^4 a \end{array}$$

Finally, $40 + a$ must be divisible by 7 and be as small as possible. So $a = 2$ and as $42 \div 7 = 6$ the units digit of n is 6.