1.	**C**
2.	**B**
3.	**D**
4.	**A**
5.	**C**
6.	**B**
7.	**B**
8.	**C**
9.	**D**
10.	**C**
11.	**E**
12.	**D**
13.	**E**
14.	**E**
15.	**C**
16.	**A**
17.	**C**
18.	**D**
19.	**E**
20.	**A**
21.	**B**
22.	**A**
23.	**E**
24.	**A**
25.	**B**

UK SENIOR MATHEMATICAL CHALLENGE

Organised by the **United Kingdom Mathematics Trust**
supported by
Institute and Faculty of Actuaries

SOLUTIONS

Keep these solutions secure until after the test on

THURSDAY 6 NOVEMBER 2014

This solutions pamphlet outlines a solution for each problem on this year's paper. We have tried to give the most straightforward approach, but the solutions presented here are not the only possible solutions. Occasionally we have added a 'Note' at the end of a solution.

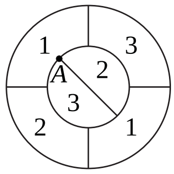
Please share these solutions with your students.

Much of the potential benefit of grappling with challenging mathematical problems depends on teachers making time for some kind of review, or follow-up, during which students may begin to see what they should have done, and how many problems they could have solved.

We hope that you and they agree that the first 15 problems could, in principle, have been solved by most candidates; if not, please let us know.

1. **C** $98 \times 102 = (100 - 2)(100 + 2) = 10000 - 4 = 9996$.

2. **B** There are points on the diagram, such as A, where the edges of three regions meet, so three or more different colours are required.
A colouring with three colours is possible as shown, so the smallest number of colours required is three.



3. **D** The year 1997 was not a leap year so had $365 = 52 \times 7 + 1$ days. Hence, starting from 1st January, 1997 had 52 complete weeks, each starting with the same day as 1st January, followed by 31st December. As 31st December was a Wednesday, so too were all the first days of the 52 complete weeks. So there were 53 Wednesdays in 1997.

4. **A** Let the original amount of money be x (in pounds). If I spend $\frac{x}{5}$ then $\frac{4x}{5}$ remains. When I spend $\frac{1}{4}$ of that, $\frac{3}{4}$ of it remains. So $\frac{4x}{5} \times \frac{3}{4}$ is what is left and that is £15. As $\frac{4x}{5} \times \frac{3}{4} = 15$, we have $x = \frac{5}{3} \times 15 = 25$. So the original amount of money is £25.

5. **C** The prime factorisations of 20 and 14 are $20 = 2 \times 2 \times 5$ and $14 = 2 \times 7$. The lowest common multiple of 20 and 14 is 140 as $140 = 2 \times 2 \times 5 \times 7$. For a number to be a multiple of 20 and 14 it must be a multiple of 140. As $2014 \div 140 = 14$ remainder 54, there are 14 integers in the required range. Note: The integer 0, which is also a multiple of 20 and of 14 is excluded as we are considering numbers *between* 1 and 2014.

6. **B** Working from right to left, the units column shows that $S = 2$ or 7. If $S = 2$, then $I + I = 1$ or 11, neither of which is possible. Hence $S = 7$ and it follows that $I + I = 0$ or 10. However, as the digits are non-zero, $I = 5$. The hundreds column then shows that $H = 9$ and so $T = 1$. This gives $T + H + I + S = 1 + 9 + 5 + 7 = 22$.

7. **B** Since $36.8 \div 86$ is approximately $40 \div 100 = 0.4$, the sea level rises by roughly 0.4 cm, which is 4 mm, per year.

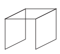
8. **C** The intersections occur in six groups and the total number of points is $2 \times 3 + 2 \times 4 + 2 \times 5 + 3 \times 4 + 3 \times 5 + 4 \times 5$ which is $6 + 8 + 10 + 12 + 15 + 20 = 71$.

9. **D** A number is divisible by 9 if and only if its digit sum is divisible by 9. The number 10^{2014} can be written as a 1 followed by 2014 zeros, so this part has a digit sum of 1. Of all the options given, only adding on 8 to this will make a digit sum of 9, so $10^{2014} + 8$ is the required answer.

10. **C** Let the length of the rectangle be x cm and its width be y cm. The area is given as $120\ \text{cm}^2$ so $xy = 120$. The perimeter is 46 cm, so $46 = 2x + 2y$ and therefore $23 = x + y$. Using Pythagoras' Theorem, the length of the diagonal is $\sqrt{x^2 + y^2}$. As $x^2 + y^2 = (x + y)^2 - 2xy$, $\sqrt{x^2 + y^2} = \sqrt{23^2 - 2 \times 120} = \sqrt{529 - 240} = \sqrt{289} = 17$. So the diagonal has length 17 cm.

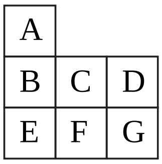
11. **E** First note that the exponent in each of the five options is prime, so we need to see which of the five numbers is not prime. By direct calculation the numbers are 3, 7, 31, 127 and 2047. Only the last number is not prime, as $2047 = 23 \times 89$.

12. **D** Let Lionel have x cherries. Michael then has $(x + 7)$ cherries. Karen's number of cherries is described in two ways. She has $3x$ cherries and also $2(x + 7)$ cherries. So $3x = 2x + 14$ and therefore $x = 14$. Lionel has 14 cherries, Michael 21 cherries and Karen 42 cherries giving a total of $14 + 21 + 42 = 77$ cherries.

13. **E** Each of P, Q, R, S and T when folded to form a cube consists of a  shape of three black faces and an interlocking  shape of three white faces, so they are all nets of the same cube.

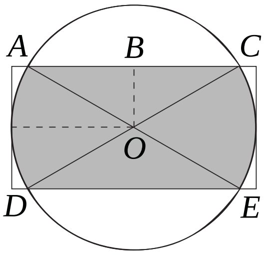
14. **E** Rearranging the equation $\dfrac{3x + y}{x - 3y} = -1$ gives $3x + y = -x + 3y$. So $4x = 2y$ and therefore $y = 2x$. Hence $\dfrac{x + 3y}{3x - y} = \dfrac{x + 3 \times 2x}{3x - 2x} = \dfrac{7x}{x} = 7$.

15. **C**



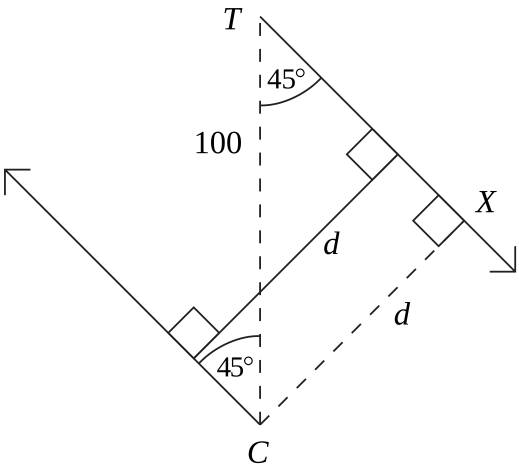
Label the squares as shown. Possible pairs to be shaded which include A are AD, AE, AF and AG. Pairs excluding A are BD, BG, DE, EG. Triples must include A and there are two possibilities, ADE and AEG. This gives 10 ways of shading the grid,

16. **A** The diameter of the circle is the same length as the longest sides of the rectangle, so the radius of the circle is 6. The perpendicular distance from the centre of the circle to the longest sides of the rectangle is half of the length of the shortest sides which is 3.



Drawing two diameters AE and DC as shown splits the shaded area into two sectors and two isosceles triangles. As OA is 6 and OB is 3, $\angle AOB = 60°$ and, by Pythagoras' Theorem, $AB = 3\sqrt{3}$. Thus $\angle AOD = 180° - 2 \times 60° = 60°$. So the shaded area is $2 \times \frac{60}{360} \times \pi \times 6^2 + 2 \times \frac{1}{2} \times 2 \times 3\sqrt{3} \times 3 = 12\pi + 18\sqrt{3}$.

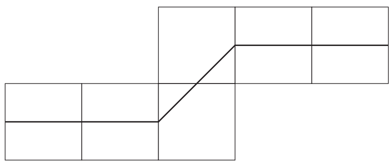
17. **C**



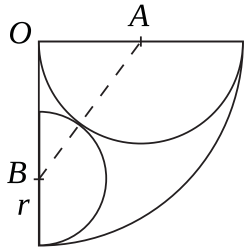
The tanker and the cruise liner are travelling in parallel and opposite directions, each making an angle of 45° with the line joining their starting positions. The shortest distance between the ships is d, the perpendicular distance between the parallel lines. This is independent of the speeds of the ships.

Considering triangle TCX gives $\sin 45° = \dfrac{d}{100}$ so $d = \dfrac{1}{\sqrt{2}} \times 100 = 50\sqrt{2}$.

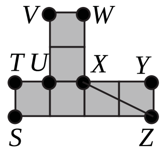
18. **D** To draw the longest unbroken line Beatrix must be able to draw her design on the net of a cube without taking her pen off the paper. She must minimise the number of lines of length $\sqrt{2}$ and maximise the number of lines of length 2. If no lines of length $\sqrt{2}$ are used, the maximum number of lines of length 2 is four, forming a loop and leaving two faces blank. Thus the longest possible unbroken line would have four lines of length 2 and two lines of length $\sqrt{2}$. A possible configuration to achieve this is shown in the diagram. The length of Beatrix's line is then $8 + 2\sqrt{2}$. Note: This path is not a loop but it is not required to be.



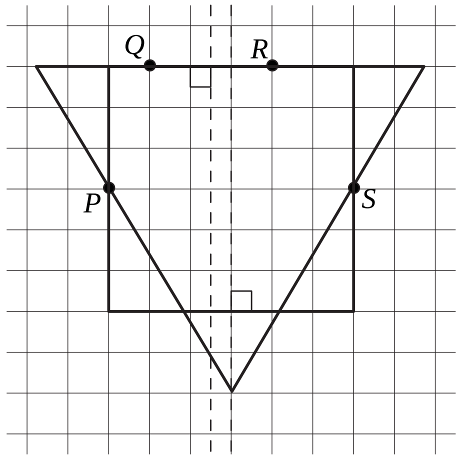
19. **E** Let the centre of the quadrant be O, the centre of the larger semicircle be A and the centre of the smaller semicircle be B. Let the radius of the smaller semicircle be r. It is given that $OA = 1$. The common tangent to the two semicircles at the point of contact makes an angle of 90° with the radius of each semicircle. Therefore the line AB passes through the point of contact, as $2 \times 90° = 180°$ and angles on a straight line sum to 180°. So the line AB has length $r + 1$. This is the hypotenuse of the right-angled triangle OAB in which $OA = 1$ and $OB = 2 - r$. By Pythagoras' Theorem $(2 - r)^2 + 1^2 = (r + 1)^2$, so $4 - 4r + r^2 + 1 = r^2 + 2r + 1$ and therefore $4 = 6r$ and so $r = \frac{2}{3}$.



20. **A** It is always possible to draw a circle through three points which are not on a straight line. The smallest circle containing all six squares must pass through (at least) three of the eight vertices of the diagram. Of all such circles, the smallest passes through S, V and Z and has its centre at X. The radius is then $\sqrt{4^2 + 2^2} = \sqrt{16 + 4} = \sqrt{20} = 2\sqrt{5}$.



21. **B** The diagram shows that it is possible to draw a square whose edges go through P, Q, R and S. By drawing lines through P and S each making an angle of 60° with QR, we can construct an equilateral triangle, as shown, whose edges pass through P, Q, R and S. However there is no circle through these four points. The centre of such a circle would be equidistant from Q and R, and hence would lie on the perpendicular bisector of QR. Similarly it would lie on the perpendicular bisector of PS, but these perpendicular bisectors are parallel lines which don't meet.



22. **A** The probability that the second marble is blue equals P(2nd marble is blue given that the 1st marble is blue) + P(2nd marble is blue given that the 1st marble is yellow), which is

$$\frac{m}{m+n} \times \frac{m+k}{m+n+k} + \frac{n}{m+n} \times \frac{m}{m+n+k} = \frac{m^2+mk+mn}{(m+n)(m+n+k)} = \frac{m(m+k+n)}{(m+n)(m+n+k)} = \frac{m}{m+n}.$$

Note: this expression is independent of k.

23. **E** If the graphs of $y = 2x$, $y = 2^x$ and $y = x^2$ are sketched on the same axes it can be seen that case (i) holds for $2 < x < 4$, case (ii) holds for $0 < x < 1$, case (iv) holds for $1 < x < 2$ and case (vi) holds for $x > 4$.

There are no real solutions for case (iii). Consider $x^2 < 2x$, which is true for $0 < x < 2$. However for $0 < x < 2$ it can be seen that $2^x > x^2$ rather than $2^x < x^2$ as stated.

There are no real solutions for case (v). Consider $2x < x^2$, which is true for $x < 0$ or $x > 2$. However, when $x < 0$ we have $2^x > 2x$ as 2^x is positive and $2x$ is negative, rather than $2^x < 2x$ as stated. Also, when $x = 2$ we have $2^x = 2x$, but for $x > 2$, $2^x > 2x$ rather than $2^x < 2x$ as stated.

24. **A** Each of the five expressions can be written in the form $\sqrt{x} - \sqrt{x-1}$, where x is in turn 100, 64, 25, 81 and 49. As $\left(\sqrt{x}-\sqrt{x-1}\right)\left(\sqrt{x}+\sqrt{x-1}\right) = x-(x-1) = 1$, we can write $\left(\sqrt{x}-\sqrt{x-1}\right) = \dfrac{1}{\left(\sqrt{x}+\sqrt{x-1}\right)}$. Since $\left(\sqrt{x}+\sqrt{x-1}\right)$ increases with x, then $\left(\sqrt{x}-\sqrt{x-1}\right)$ must decrease with x. Therefore, of the given expressions, the one corresponding to the largest value of x is the smallest. This is $\sqrt{100} - \sqrt{99}$ which is $10 - 3\sqrt{11}$.

25. **B** Let the supplementary angle to α be β. Let tile 1 on the outside of the star polygon be horizontal. Counting anti-clockwise around the star polygon, tile 3 has an angle of elevation from the horizontal of $\beta - \alpha = 96\tfrac{2}{3}° - 83\tfrac{1}{3}° = 13\tfrac{1}{3}°$. As $360° \div 13\tfrac{1}{3}° = 27$, we need 27 pairs of tiles to complete one revolution. So there are 54 tiles in the complete pattern.