1.	**A**
2.	**C**
3.	**D**
4.	**C**
5.	**E**
6.	**E**
7.	**B**
8.	**B**
9.	**A**
10.	**B**
11.	**A**
12.	**B**
13.	**D**
14.	**D**
15.	**B**
16.	**D**
17.	**A**
18.	**C**
19.	**C**
20.	**E**
21.	**B**
22.	**C**
23.	**A**
24.	**E**
25.	**D**

UK SENIOR MATHEMATICAL CHALLENGE

Organised by the **United Kingdom Mathematics Trust**

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Institute and Faculty of Actuaries

SOLUTIONS

Keep these solutions secure until after the test on

THURSDAY 7 NOVEMBER 2013

This solutions pamphlet outlines a solution for each problem on this year's paper. We have tried to give the most straightforward approach, but the solutions presented here are not the only possible solutions. Occasionally we have added a 'Note' (in italics).

Please share these solutions with your students.

Much of the potential benefit of grappling with challenging mathematical problems depends on teachers making time for some kind of review, or follow-up, during which students may begin to see what they should have done, and how many problems they could have solved.

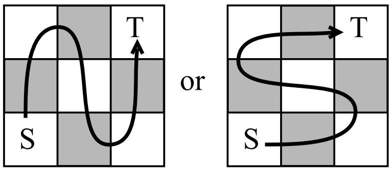
We hope that you and they agree that the first 15 problems could, in principle, have been solved by most candidates; if not, please let us know.

1. **A** Calculating the value of each option gives $2 + 0 + 1 + 3 = 6$, $2 \times 0 + 1 + 3 = 4$, $2 + 0 \times 1 + 3 = 5$, $2 + 0 + 1 \times 3 = 5$ and $2 \times 0 \times 1 \times 3 = 0$ so $2 + 0 + 1 + 3$ is the largest.

2. **C** In metres, the height 2m 8cm and 3mm is $2 + 8 \times 0.01 + 3 \times 0.001 = 2 + 0.08 + 0.003 = 2.083$ m.

3. **D** Factorising $2013^2 - 2013$ gives $2013(2013 - 1)$ which equals 2013×2012. So the tens digit is 5 as $13 \times 12 = 156$ and only this part of the product contributes to the tens digit of the answer.

4. **C** In order to pass through each square exactly once, a route must pass in and out of both unlabelled corner squares and also pass through the middle. Passing in and out of a corner involves three squares, coloured grey, white and grey in that order. Passing in and out of the two unlabelled corners therefore accounts for six unlabelled squares, leaving only the middle square which must be in the middle of any possible route. So, there are two possible routes as shown.



5. **E** Since $x(y + 2) = 100$ and $y(x + 2) = 60$ then $xy + 2x = 100$ and $xy + 2y = 60$. Subtracting gives $2x - 2y = 40$ and therefore $x - y = 20$.

6. **E** Let d be the number of lengths that Rebecca intended to swim. Then $6 = \frac{d}{4} - \frac{d}{5} = \frac{d}{20}$ and therefore $d = 6 \times 20 = 120$.

7. **B** The first item that Susanna buys makes her bill a number of pounds and 99 pence. Each extra item she buys after that decreases by one the number of pence in her total bill. Let n be the number of items bought. To be charged £65.76, $1 + 99 - n = 76$ so $n = 100 - 76 = 24$. Alternatives of 124 items or more are infeasible as they would each give a total greater than £65.76.

8. **B** The area of the shaded square is equal to the area of the large square minus the area of the four triangles. Thus the area of the shaded square is $(4x)^2 - 4 \times \frac{1}{2} \times 4x \times x = 16x^2 - 8x^2 = 8x^2$. So the side-length is $\sqrt{8x^2} = 2\sqrt{2}x$.

9. **A** When a square loses a quarter of its area, thereby becoming a smaller square, three quarters of its area remains. Therefore the lengths of the sides of the original square have been multiplied by $\sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3} \approx 0.866$. This means a reduction of $(100 - 86.6)\%$ which is approximately 13%.

10. **B** The median is 10. Therefore the mode must be 11 and there must be two 11s in Frank's list. The mean is 9, so the total of the five numbers is 45. This means that the total of the two smallest integers is $45 - (10 + 2 \times 11) = 13$. The maximum size of the second largest integer is 9 so the smallest integer that Frank could include in his list is $13 - 9 = 4$.

11. **A** Let the radius of the circle be r. Then its area is πr^2. The height of the triangle is r and its area is $\frac{1}{2} \times PQ \times r$. So $\frac{1}{2} \times PQ \times r = \pi r^2$ and therefore $PQ = 2\pi r$, which is also the circumference of the circle. Therefore the ratio of the length of PQ to the circumference of the circle is 1 : 1.

12. **B** There are three options for Sammy's first choice and then two options for each subsequent choice. Therefore the number of possible ways is $3 \times 2 \times 2 \times 2 \times 2 = 48$.

13. **D** Angus completes the course in 40 minutes, so he spends 20 minutes (which is $\frac{1}{3}$ of an hour) walking and the same time running. By using distance = speed × time, the length of the course is $3 \times \frac{1}{3} + 6 \times \frac{1}{3} = 1 + 2 = 3$ miles.
Bruce completes the course by walking for $1\frac{1}{2}$ miles and running for $1\frac{1}{2}$ miles. So, by using time $= \frac{\text{distance}}{\text{speed}}$, Bruce's total time in hours is $\frac{1\frac{1}{2}}{3} + \frac{1\frac{1}{2}}{6} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ of an hour. So Bruce takes 45 minutes to complete the course.

14. **D** Triangle RST is similar to triangle RPS as their corresponding angles are equal. Using Pythagoras' Theorem, the ratio of RS to RP is $1 : \sqrt{5}$. So the ratio of RT to RS is also $1 : \sqrt{5}$. Therefore the ratio of the area of the triangle RST to the area of triangle RPS is $1 : 5$. Triangle RPS is half the rectangle $PQRS$, so the ratio of the area of triangle RST to the area of rectangle $PQRS$ is $1 : 10$.

15. **B** A prime number has exactly two factors, one of which is 1. The expression $4^n - 1$ can be factorised as $4^n - 1 = (2^n + 1)(2^n - 1)$. For $4^n - 1$ to be prime, the smaller of the factors, $2^n - 1$, must equal 1.
If $2^n - 1 = 1$ then $2^n = 2$ giving $n = 1$. So there is exactly one value of n for which $4^n - 1$ is prime and this value is 1.

16. **D** By the Fundamental Theorem of Arithmetic, every positive integer greater than 1 is either prime or a product of two or more primes. A number that is the product of two or more primes is called a *composite* number.
We are looking to choose, from the options provided, a composite number which is of the form $8n + 3$ but does not have a prime factor of the form $8n + 3$.
Option A is prime, so is not possible. Options B and C are not of the form $8n + 3$.
Option E is $8 \times 12 + 3 = 99$. The number 99, when expressed as a product of its prime factors, is $3 \times 3 \times 11$ and the factor 11 is of the required form as $11 = 8 \times 1 + 3$.
However, option D is of the form $8n + 3$ as $8 \times 11 + 3 = 91$ but neither of the prime factors of 91, which are 7 and 13, are of the form $8n + 3$.

17. **A** Triangle PQR is equilateral so $\angle QPU = \angle UPT = \angle TPR = 20°$. Triangle PUT is isosceles, so $\angle PUT = 80°$. Let X be the midpoint of PQ and Y be the midpoint of UT.
Considering the right-angled triangle PXU gives $\cos 20° = \dfrac{PX}{PU} = \dfrac{\frac{1}{2}}{PU}$, so $PU = \dfrac{1}{2\cos 20°}$.
Considering the right-angled triangle PUY gives $\cos 80° = \dfrac{UY}{PU}$, so $UY = PU \cos 80° = \dfrac{\cos 80°}{2\cos 20°}$. Therefore $UT = 2UY = \dfrac{2\cos 80°}{2\cos 20°} = \dfrac{\cos 80°}{\cos 20°}$.
{*Note that triangle UTS is a Morley triangle, named after the mathematician Frank Morley. His 1899 trisector theorem states that in any triangle, the three points of intersection of the adjacent angle trisectors form an equilateral triangle, in this case, triangle UTS.*}

18. **C** The product of all the numbers in the list is $2 \times 3 \times 12 \times 14 \times 15 \times 20 \times 21$ which, when expressed in terms of prime factors is $2 \times 3 \times 2 \times 2 \times 3 \times 2 \times 7 \times 3 \times 5 \times 2 \times 2 \times 5 \times 3 \times 7$ which is equal to $2^6 \times 3^4 \times 5^2 \times 7^2 = (2^3 \times 3^2 \times 5 \times 7)^2 = 2520^2$. The answer 2520 is expressible as both $2 \times 3 \times 20 \times 21$ and $12 \times 14 \times 15$.

19. **C** There are 25 vertices in the diagram. Each vertex is part of a row of 5 vertices and a column of 5 vertices. Each vertex is therefore an integer number of units away from the 4 other vertices in its row and from the other 4 vertices in its column. This appears to give $25 \times (4 + 4) = 200$ pairs. However, counting in this manner includes each pair twice so there are only 100 different pairs.
By using the Pythagorean triple 3, 4, 5, each corner vertex is five units away from two other non-corner vertices, giving another 8 pairs. No other Pythagorean triples include small enough numbers to yield pairs of vertices on this grid.
Thus the total number of pairs is 108.

20. **E** Let the two positive numbers be x and y with $x > y$. The sum of the numbers is greater than their difference, so the two ratios which are equal are $x : y$ and $x + y : x - y$. Therefore $\dfrac{x}{y} = \dfrac{x+y}{x-y}$. By dividing the top and bottom of the right-hand side by y we obtain $\dfrac{x}{y} = \dfrac{\frac{x}{y} + 1}{\frac{x}{y} - 1}$.
Letting $k = \frac{x}{y}$ gives $k = \frac{k+1}{k-1}$ which gives the quadratic $k^2 - 2k - 1 = 0$. Completing the square gives $(k-1)^2 = 2$ whence $k = 1 \pm \sqrt{2}$. However, as x and y are both positive, $k \neq 1 - \sqrt{2}$. As the ratio $\frac{x}{y} = 1 + \sqrt{2}$, the ratio $x : y$ is $1 + \sqrt{2} : 1$.

21. **B** Let the top vertex of the square be A and the midpoints of the two lines that meet at A be B and C. The line BC is of length $\frac{1}{2}$ and is perpendicular to the diagonal of the square through A. Let the point of intersection of these two lines be D. Let the end of the uppermost arc, above B, be E. Then $ADBE$ is a rhombus, made from four radii of the arcs, AD, DB, BE and EA, each of length $\frac{1}{4}$. As $\angle ADB = 90°$, this rhombus is a square.

It then follows that the four arcs whose centres are the vertices of the original square are all semi-circles. The remaining four touching arcs are each $\frac{3}{4}$ of a circle.

In total, the length of the border is $4 \times \frac{1}{2} + 4 \times \frac{3}{4}$ times the circumference of a circle with the same radius, so is $5 \times 2\pi \times \frac{1}{4} = \frac{5}{2}\pi$.

22. **C** The numbers in the sequence 11, 21, 31, 41, …. , 981, 991 are of the form $10n + 1$ for $n = 1$ to 99. There are therefore 99 numbers in this sequence.

Twelve terms of this sequence can be expressed using factors of the form $10k + 1$. In this form, these terms are 11×11, 11×21, 11×31, …, 11×81 and 21×21, 21×31, 21×41 and 31×31. All other pairings give products that are too large.

Hence, there are $99 - 12 = 87$ 'grime' numbers.

23. **A** The pentagon $RTWVU$ is the remainder when triangles SUV and WTQ are removed from the bottom right half of the square. Draw in the diagonal PR and consider the triangle PRS. The medians of triangle PRS join each vertex P, R and S to the midpoint of its opposite side, i.e. P to U and S to the middle of the square. The medians intersect at V and therefore the height of V above SR is $\frac{1}{3}$ of PS.

The area of triangle SUV is therefore $\frac{1}{2} \times \frac{1}{2}SR \times \frac{1}{3}PS = \frac{1}{12}$ of the area of the square. By symmetry, this is also the area of triangle WTQ. The area of the pentagon $RTWVU$ is then $\frac{1}{2} - \left(\frac{1}{12} + \frac{1}{12}\right) = \frac{1}{3}$ of the area of the square $PQRS$.

24. **E** As they are vertically opposite, $\angle POQ = \angle SOR$. Let α denote the size of each of these.

Applying the cosine rule to triangle SOR gives $8^2 = 4^2 + 5^2 - 2 \times 4 \times 5 \cos\alpha$, therefore $40 \cos\alpha = -23$.

Similarly, from triangle POQ we obtain $x^2 = 4^2 + 10^2 - 2 \times 4 \times 10 \cos\alpha$. So $x^2 = 16 + 100 - 2 \times (-23) = 162$.

Hence $x = \sqrt{162} = \sqrt{81 \times 2} = 9\sqrt{2}$.

25. **D** Jessica must travel alternately on lines which are connected to station X (i.e. s, t or u), and connected to station Y (i.e. p, q or r). In order to depart from X and end her journey at Y, she must travel along an even number of lines. This can be 2, 4 or 6 lines, making 1, 3 or 5 changes respectively.

Case A, 2 lines: Jessica leaves station X along one of the lines s, t or u, makes one change onto one of lines p, q or r and reaches station Y. Here there are 3×3 possibilities.

Case B, 4 lines: Jessica leaves station X along one of the lines s, t or u and makes her first change onto one of lines p, q or r. She then makes her second change onto either of the two lines s, t or u on which she has not previously travelled and her third change onto either of the two lines p, q or r on which she has not previously travelled and reaches station Y. Here there are $3 \times 3 \times 2 \times 2$ possibilities.

Case C, 6 lines: Her journey is as described in Case B but her fourth change is onto the last of the lines s, t or u on which she has not previously travelled and her fifth change is onto the last of the lines p, q or r on which she has not previously travelled. Here there are $3 \times 3 \times 2 \times 2 \times 1 \times 1$ possibilities.

So in total Jessica can choose $9 + 36 + 36 = 81$ different routes.