1.	C
2.	B
3.	C
4.	B
5.	D
6.	B
7.	C
8.	D
9.	D
10.	E
11.	D
12.	A
13.	A
14.	E
15.	E
16.	E
17.	E
18.	A
19.	B
20.	C
21.	A
22.	B
23.	B
24.	E
25.	D

UK SENIOR MATHEMATICAL CHALLENGE

Organised by the **United Kingdom Mathematics Trust**
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The Actuarial Profession
making financial sense of the future

SOLUTIONS

Keep these solutions secure until after the test on

THURSDAY 4 NOVEMBER 2010

This solutions pamphlet outlines a solution for each problem on this year's paper. We have tried to give the most straightforward approach, but the solutions presented here are not the only possible solutions. Occasionally we have added a 'Note' (in italics).

Please share these solutions with your students.

Much of the potential benefit of grappling with challenging mathematical problems depends on teachers making time for some kind of review, or follow-up, during which students may begin to see what they should have done, and how many problems they could have solved.

We hope that you and they agree that the first 15 problems could, in principle, have been solved by most candidates; if not, please let us know.

1. **C** The only two-digit cubes are 27 and 64. As 1 *Down* is one less than a cube then 3 *Across* must start with 6 or 3 and so is 64. Thus $x = 4$.

2. **B** The smallest possible value is attained by using $p = 1$, $q = 2$ and $r = 3$. Therefore this value is $20 \times 1 + 10 \times 2 + 3 = 43$.

3. **C** The three internal angles of an equilateral triangle are all 60°. As the sum of the angles on a straight line is 180° then the sum of the four marked angles is $2 \times (180 - 60)° = 2 \times 120° = 240°$.

4. **B** $2 + 0 + 1 + 1 = 4$. Multiples of 4 are even, hence 2011 is not valid and the same argument applies to 2013, 2015, 2017 and 2019.
$2 + 0 + 1 + 2 = 5$. The units digit for multiples of 5 is 0 or 5, hence 2012 is not valid.
$2 + 0 + 1 + 4 = 7$. But $\frac{2014}{7} = 287\frac{5}{7}$, hence 2014 is not valid.
$2 + 0 + 1 + 6 = 9$. Since $2016 = 9 \times 224$, 2016 is valid.
Hence we have to wait $2016 - 2010 (= 6)$ more years.

5. **D** If the statement is true then the capacity (in litres) of Morecambe Bay is approximately:

$$20 \times 10^6 \times 365 \times 24 \times 60 \times 6 = 10^8 \times (6 \times 365) \times (2 \times 24) \times 6$$
$$\approx 6 \times 10^8 \times 2000 \times 50 = 6 \times 10^{13}.$$

6. **B** The length of the road is 8km. Hence the time taken to run down the mountain is $\frac{8}{12}$ hours $= \frac{8}{12} \times 60$ min = 40 min.

7. **C** There are 24 arrangements of the letters in the word ANGLE with A as the first letter. In alphabetical order AEGLN is first and ANLGE is last ie 24th. ANLEG is the 23rd and hence ANGLE is the 22nd.

8. **D** $(x + y + z)(x - y - z) = [x + (y + z)][x - (y + z)] = x^2 - (y + z)^2$.

9. **D** $(2 \diamond 3) \diamond 4 = (2^3 - 3^2) \diamond 4 = (-1) \diamond 4 = (-1)^4 - 4^{-1} = 1 - \frac{1}{4} = \frac{3}{4}$.

10. **E** Let the original square have sides of length y cm and the single square which is not 1×1 have sides of length x cm. Then $y^2 = 36 + x^2$, and so $y^2 - x^2 = 36$ and hence $(y + x)(y - x) = 36$.
As $36 = 2^2 \times 3^2$ and $y + x > y - x$ the possible factors of 36 are:

$y + x$	$y - x$	y	x	
9	4	$6\frac{1}{2}$	$2\frac{1}{2}$	impossible
12	3	$7\frac{1}{2}$	$4\frac{1}{2}$	impossible
18	2	10	8	possible
36	1	$18\frac{1}{2}$	$17\frac{1}{2}$	impossible

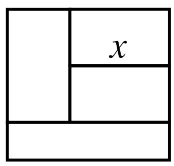
We can check that $10^2 = 36 + 8^2 = 100$ and hence the length of the side of the *original* square is 10 cm.

11. **D** Squaring the numbers given allows us to see their order easily:

$(9\sqrt{2})^2 = 81 \times 2 = 162$ $\quad$ $(3\sqrt{19})^2 = 9 \times 19 = 171$ $\quad$ $(4\sqrt{11})^2 = 16 \times 11 = 176$
$(5\sqrt{7})^2 = 25 \times 7 = 175$ $\quad$ $(6\sqrt{5})^2 = 36 \times 5 = 180$

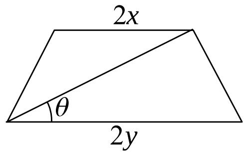
As 175 is the middle one of these numbers, the answer is $5\sqrt{7}$.

12. **A** As the square has side length 1 its area is $1 \times 1 = 1$.
Thus the area of each of the four rectangles is $\frac{1}{4}$.
The length of the bottom rectangle is 1 hence its width is $\frac{1}{4}$.
Thus the width of each of the two congruent rectangles is $\frac{1}{2}(1 - \frac{1}{4}) = \frac{3}{8}$.
Hence the area of one of these congruent rectangles is $\frac{3}{8}x$.
But we know this area is $\frac{1}{4}$, therefore $\frac{3}{8}x = \frac{1}{4}$ and hence $x = \frac{2}{3}$.



13. A The lowest common multiple of 3 and 4 is 12. Hence both of the required conditions are satisfied only by numbers that are 2 less than multiples of 12 and also less than 100, ie: 10, 22, 34, 46, 58, 70, 82 and 94.
Therefore 8 two-digit numbers satisfy the conditions.

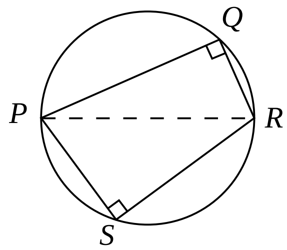
14. E Drop perpendiculars from the top vertices to the bottom line. The distance from the foot to the nearer base vertex is $\frac{1}{2}(2y - 2x) = y - x$. So the distance to the further base vertex is $2y - (y - x) = y + x$.



Hence $\cos\theta = \dfrac{x + y}{d}$ where d is the length of the diagonal.

15. E The first eight prime numbers are: 2, 3, 5, 7, 11, 13, 17, 19.
If the sum of two prime numbers is prime, one of them must be 2.
If the sum of three different prime numbers is prime they must all be odd. The answer is therefore 19 as: $2 + 17 = 19$ and $3 + 5 + 11 = 19$.

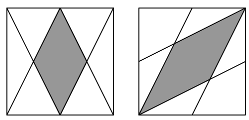
16. E As PR is a diameter, $\angle PQR = \angle PSR = 90°$ (angles in a semicircle are 90°).
Since $PQ = 12 \times 5$ and $QR = 5 \times 5$, triangle PQR is an enlarged 5, 12, 13 triangle and so $PR = 13 \times 5 = 65$.
Since $PR = 5 \times 13$ and $SR = 4 \times 13$, triangle PRS is an enlarged 3, 4, 5 triangle and so $SP = 3 \times 13 = 39$.



17. E $\sqrt{9^{16x^2}} = 9^{(16x^2)/2} = 9^{8x^2}$.

18. A Let x be the length of the shaded rectangle.
By Pythagoras' Theorem, $x^2 = 2^2 + 2^2$, hence $x = 2\sqrt{2}$.
The total surface area of the two prisms equals the surface area of the solid cube plus twice the surface area of that shaded rectangle, that is $6 \times 2 \times 2 + 2 \times 2 \times 2\sqrt{2} = 24 + 8\sqrt{2} = 8(3 + \sqrt{2})$.

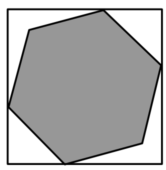
19. B In the rhombus on the left, drawing vertical straight lines at distances of $1\frac{1}{2}$, 3 and $4\frac{1}{2}$ from the left edge of the square, and a horizontal straight line bisecting the square, creates 16 equivalent triangles. Of these, four are shaded giving a total shaded area of $\frac{1}{4} \times 6 \times 6 = 9$.



Draw in the diagonal from NW to SE in the rhombus on the right. The four unshaded triangles now above the shaded area are all equal in area (a say); and one can see that 3 of these together make up $\frac{1}{4}$ of the square. Hence $a = 3$. Thus the shaded area equals $36 - 3 \times 8 = 12$.
Therefore the difference between the shaded areas is $12 - 9 = 3$.

20. C Let the number of boys in the class be x. Hence $\dfrac{10}{10 + x} \times \dfrac{9}{9 + x} = \dfrac{3}{20}$.
Simplifying gives $1800 = 3(10 + x)(9 + x)$ and then $x^2 + 19x - 510 = 0$.
Factorising gives $(x + 34)(x - 15) = 0$ and, since $x \neq -34$, $x = 15$.

21. A The hypotenuse of one of the small right-angled triangles is parallel to the diagonal and hence makes angles of 45°. Since the hypotenuse has length 1, the other two sides have length $\frac{1}{\sqrt{2}}$, by Pythagoras' Theorem.
As the internal angle of a regular hexagon is 120°, drawing a diagonal from NW to SE forms two triangles, bottom right, each with angles 45°, 120° and 15°. (The sum of the angles in a triangle is 180°).



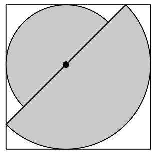
Let the square have length y units. Using the Sine Rule gives $\dfrac{y - \frac{1}{\sqrt{2}}}{\sin 120°} = \dfrac{1}{\sin 45°}$.

Hence $y - \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{3}}{2} \div \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{3}}{\sqrt{2}}$ and therefore $y = \dfrac{\sqrt{3} + 1}{\sqrt{2}}$.

Hence the area of the square is $y^2 = \left(\dfrac{\sqrt{3} + 1}{\sqrt{2}}\right)^2 = \dfrac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3}$.

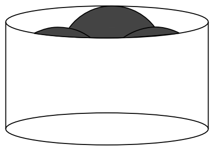
22. **B** Since $x^2 - px - q = 0$, it follows that $x^3 = px^2 + qx$.
But $x^2 = px + q$ and so $x^3 = p(px + q) + qx$, ie $x^3 = (p^2 + q)x + pq$.
The three possible values shown for pq are 3, 5 and 7.
If $pq = 3$, $p^2 + q = 1^2 + 3 = 4$ or $p^2 + q = 3^2 + 1 = 10$. Hence $4x + 3$ and $10x + 3$ could equal x^3.
If $pq = 7$, we may take $p = 1, q = 7$ to get $p^2 + q = 1^2 + 7 = 8$. Hence $8x + 7$ could equal x^3.
If $pq = 5$, we may take $p = 5, q = 1$ to get $p^2 + q = 5^2 + 1 = 26$. Hence $26x + 5$ could equal x^3.
However, the only other possibility, $p = 1, q = 5$ gives $p^2 + q = 6 \neq 8$. Therefore $8x + 5 \neq x^3$.

23. **B** Let r_1 and r_2 represent the radii of the smaller and larger semicircles respectively. A vertical line through the common centre of the semicircles gives $r_1 + r_2 = 2 \ldots (1)$. Also, together with the diameter of the larger semicircle, this line forms a right-angled, isosceles triangle giving $\sin 45° = \frac{r_1}{r_2}$. Hence $r_2 = \sqrt{2}r_1 \ldots (2)$.



Substituting (2) into (1) gives $(1 + \sqrt{2})r_1 = 2$ so that $r_1 = 2(\sqrt{2} - 1)$.
Therefore $r_2 = 2\sqrt{2}(\sqrt{2} - 1)$.
Hence the total shaded area is $\frac{1}{2}\pi(r_1^2 + r_2^2) = \frac{1}{2}\pi\left[4(\sqrt{2}-1)^2 + 8(\sqrt{2}-1)^2\right] = 6\pi(3 - 2\sqrt{2})$.

24. **E** The volume of the three spheres is $3 \times \frac{4}{3}\pi \times 1^3 = 4\pi$.
Let r be the radius of the cross-sectional area of the cylinder.
Hence the volume of the cylinder is $2\pi r^2$.
Thus the required fraction is $\frac{2}{r^2}$.



The straight lines joining the centres of the three spheres form an equilateral triangle of side length 2.
Let x be the distance from the centre of a sphere to the midpoint of the triangle. Using the Sine Rule, $\frac{2}{\sin 120°} = \frac{x}{\sin 30°}$ hence $x = \frac{2}{\sqrt{3}}$.
As the sphere has radius 1, $r = x + 1$ and $r = 1 + \frac{2}{\sqrt{3}}$.
Thus $r^2 = \frac{1}{3}(2 + \sqrt{3})^2 = \frac{1}{3}(7 + 4\sqrt{3})$. Hence the required fraction is $\frac{6}{7 + 4\sqrt{3}}$.

25. **D** The sum of 10 different digits is 45. As the sum of the digits in the question is 36 then digits adding to 9 are omitted.
The combinations of digits satisfying this are:

$$9;\ 1 + 8;\ 2 + 7;\ 3 + 6;\ 4 + 5;\ 1 + 2 + 6;\ 1 + 3 + 5;\ 2 + 3 + 4.$$

When '0' is not involved there are $(8! + 4 \times 7! + 3 \times 6!)$ numbers, whereas when '0' is used there are $(8 \times 8! + 4 \times 7 \times 7! + 3 \times 6 \times 6!)$.
This gives a total of $9 \times 8! + (4 + 28) \times 7! + (3 + 18) \times 6! = (72 + 32 + 3) \times 7! = 107 \times 7!$
Hence $N = 107$.