1.	A
2.	D
3.	E
4.	E
5.	A
6.	C
7.	B
8.	C
9.	D
10.	B
11.	C
12.	E
13.	C
14.	D
15.	D
16.	B
17.	C
18.	A
19.	C
20.	E
21.	B
22.	C
23.	D
24.	B
25.	A

UK SENIOR MATHEMATICAL CHALLENGE

Organised by the **United Kingdom Mathematics Trust**

SOLUTIONS

Keep these solutions secure until after the test on

THURSDAY 5 NOVEMBER 2009

This solutions pamphlet outlines a solution for each problem on this year's paper. We have tried to give the most straightforward approach, but the solutions presented here are not the only possible solutions. Occasionally we have added a 'Note' (in italics).

Please share these solutions with your students.

Much of the potential benefit of grappling with challenging mathematical problems depends on teachers making time for some kind of review, or follow-up, during which students may begin to see what they should have done, and how many problems they could have solved.

We hope that you and they agree that the first 15 problems could, in principle, have been solved by most candidates; if not, please let us know.

The UKMT is a registered charity.

1. **A** 20% of 30% = $0.2 \times 0.3 = 0.06 = 6\%$.

2. **D** $\dfrac{785}{15} = 52\tfrac{1}{3}$ hence 785 is not a multiple of 15. But $\dfrac{135}{15} = 9$, $\dfrac{315}{15} = 21$, $\dfrac{555}{15} = 37$, $\dfrac{915}{15} = 61$.

3. **E** $1 - 32 + 81 - 64 + 25 - 6 = 5$.

4. **E** Steve achieved $\dfrac{150}{10} \times 4.5$ miles per gallon which is $15 \times 4.5 = 67.5 \approx 70$.

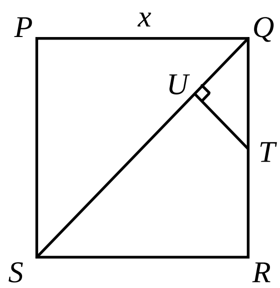
5. **A** As the ratio of the radii is 3 : 4 then the number of revolutions made by the larger wheel is $120000 \times \tfrac{3}{4} = 90000$.

6. **C** If at most two marbles of each colour are chosen, the maximum number we can choose is 8, corresponding to 2 of each. Therefore, if 9 are chosen, we must have at least 3 of one colour, but this statement is not true if 9 is replaced by any number less than 9.

7. **B** The top left 2 by 2 outlined block must contain a 3 and a 4 and this can be done in two ways. For each choice there is only one way to complete the entire mini-sudoku.

8. **C** The increase in entries from 2007 to 2008 is $92\,690 - 87\,400 = 5290$.
Hence the percentage increase is $\dfrac{5290}{87400} \times 100\% = \dfrac{5290}{874}\% \approx \dfrac{5400}{900}\% = 6\%$.
(The exact value is $6\tfrac{1}{19}$.)

9. **D** As T is the midpoint of QR then $QT = \tfrac{1}{2}x$.
Since $\angle UQT = \angle SQR = 45°$ and $\angle QUT = 90°$, $\angle UTQ = 45°$.
Thus triangle QTU is isosceles with $UQ = UT$.
In triangle QTU, by Pythagoras' Theorem, $QT^2 = QU^2 + TU^2$.
Hence $(\tfrac{1}{2}x)^2 = 2TU^2$ so $TU^2 = \tfrac{1}{8}x^2$ giving $TU = \dfrac{x}{2\sqrt{2}}$.



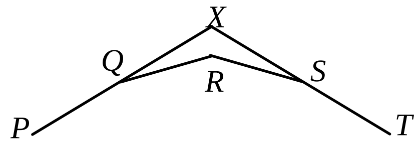
10. **B** A number is a multiple of 6 precisely when it is both a multiple of 2 and of 3. To be a multiple of 2, it will need to end with an even digit; i.e. 0 or 2. If it ends with 0, the sum of the other two digits must be a multiple of 3; and only $3 = 1 + 2$ or $6 = 1 + 5$ are possible. That gives the numbers 120, 210, 150, 510. If it ends with 2, the sum of the others must be $1 = 0 + 1$ or $4 = 1 + 3$. That gives 102, 132 and 312.

11. **C** $\sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} = 4\sqrt{2} = 2^2 \times 2^{1/2} = 2^{2\frac{1}{2}}$. Hence $x = 2\tfrac{1}{2}$.

12. **E** $\cos 50° < \sin 50° < 1$. Hence $\dfrac{1}{\cos 50°} > \dfrac{1}{\sin 50°} > 1 > \sin 50° > \cos 50°$.
$\tan 50° = \dfrac{\sin 50°}{\cos 50°} < \dfrac{1}{\cos 50°}$ hence $\dfrac{1}{\cos 50°}$ has the greatest value.

13. **C** $x - \dfrac{1}{x} = y - \dfrac{1}{y}$ hence $x^2y - y = xy^2 - x$. Thus $xy(y - x) + y - x = 0$.
Therefore $(y - x)(xy + 1) = 0$. As $x \neq y$ then $y - x \neq 0$.
Hence $xy + 1 = 0$ giving $xy = -1$.

14. **D** Let the external angle of the regular polygon be $x°$.
Hence $\angle XQR = \angle XSR = x°$ and reflex angle $\angle QRS = (180 + x)°$.
As the sum of the angles in the quadrilateral $QRSX$ is $360°$ then $140 + x + x + 180 + x = 360$.
Hence $3x = 40$ and the polygon has $\dfrac{360}{40 \div 3} = 27$ sides.



15. **D** Let $\dfrac{n}{100 - n} = x$ where x is an integer. Hence $n = 100x - nx$.
Hence $n(1 + x) = 100x$ giving $n = \dfrac{100x}{1 + x}$.
Now x and $1 + x$ can have no common factors. Therefore $1 + x$ must be a factor of 100 and can be any of them.
Hence $1 + x \in \{\pm1, \pm2, \pm4, \pm5, \pm10, \pm20, \pm25, \pm50, \pm100\}$ thus the number of possible integers n is 18.

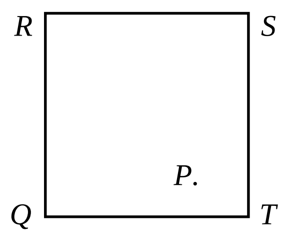
16. **B** Since $x^4 - y^4 = 2009$ it follows that $(x^2 + y^2)(x^2 - y^2) = 2009$.
But $x^2 + y^2 = 49$ hence $x^2 - y^2 = \dfrac{2009}{49} = 41$.
Subtracting gives $2y^2 = 8$ hence $y^2 = 4$. Since $y > 0, y = 2$.

17. **C** The greatest possible value of f is achieved by a rectangular cut through an edge of a cube and the furthest edge from it. If we take x as the side of the cube, by Pythagoras' Theorem the extra surface area formed by the cut is $2\sqrt{2}x^2$. Hence $f = \dfrac{2\sqrt{2}x^2}{6x^2} = \dfrac{\sqrt{2}}{3}$.

18. **A** We have $y^2 = x(2 - x)$. Now $y^2 \geqslant 0$ for all real y hence $x(2 - x) \geqslant 0$.
Hence $0 \leqslant x \leqslant 2$. In fact we can rewrite the equation as $(x - 1)^2 + y^2 = 1$; so this is a circle of radius 1 with centre (1,0).

19. **C** The distance cycled by Hamish between noon and 4 pm is $4x$.
The distance cycled by Ben between 2 pm and 4 pm is $2y$.
They meet at 4 pm hence $4x + 2y = 51$ or $2x + 2(x + y) = 51$ (*).
If they had both started at noon then they would have met at 2:50 pm and so $2\frac{5}{6}(x + y) = 51$.
Hence $x + y = 51 \times \dfrac{6}{17} = 18$. Hence from (*) $2x + 2 \times 18 = 51$.
Hence $2x = 15$ giving $x = 7\frac{1}{2}$. Thus $y = 10\frac{1}{2}$.

20. **E** If $\angle RPQ = 90°$ then P lies on a semicircle of diameter RQ.
Let x be the side-length of the square $QRST$.
Hence the area of the semicircle $RPQ = \frac{1}{2}\pi\left(\frac{1}{2}x\right)^2 = \frac{1}{8}\pi x^2$ and the area of square $QRST$ is x^2.
$\angle RPQ$ is acute when P is outside the semicircle RPQ.



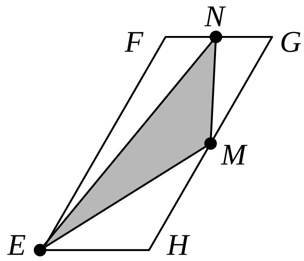
Hence the probability that $\angle RPQ$ is acute is $\dfrac{x^2 - \frac{1}{8}\pi x^2}{x^2} = 1 - \dfrac{\pi}{8}$.

21. **B** Let r be the radius of the small cone and h the height.
Let l_1 and l_2 be the slant heights of the small and large cones respectively.
By Pythagoras' Theorem $l_2 = \sqrt{6^2 + 8^2} = 10$.
Using similar triangles, $\frac{l_1}{r} = \frac{10}{6}$ so $l_1 = \frac{5}{3}r$ and $\frac{h}{8} = \frac{r}{6}$ giving $h = \frac{4}{3}r$.
Thus the area of the curved surface of the frustum is

$$\pi \times 6 \times 10 - \pi \times r \times \frac{5}{3} \times r = \pi\left(60 - \frac{5r^2}{3}\right).$$

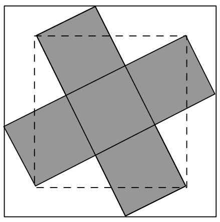
The sum of the areas of the two circles is $\pi \times 6^2 + \pi \times r^2 = \pi(36 + r^2)$.
Hence $\pi\left(60 - \frac{5r^2}{3}\right) = \pi(36 + r^2)$ and so $24 = \frac{8r^2}{3}$ giving $r = 3$, so $h = \frac{4}{3} \times 3 = 4$.
Therefore, in cms, the height of the frustum is $8 - 4 = 4$.

22. **C** Let the perpendicular distance between EH and FG be x cm and the area of the parallelogram $EFGH$ be y cm^2. Thus $y = FG \times x$.
The area of triangle EFN is $\frac{1}{2}FN \times x = \frac{1}{2} \times \frac{1}{2} \times FG \times x = \frac{1}{4}y\,\text{cm}^2$.
Likewise the areas of triangles EHM and NGM are $\frac{1}{4}y$ cm^2 and $\frac{1}{8}y$ cm^2 respectively.
The area of triangle ENM is 12 cm^2, hence $y = 12 + \frac{5}{8}y$ and so $y = 32$. Hence the area of the parallelogram $EFGH$ is 32 cm^2.



23. **D** Label the rows of the triangles from left to right as follows: $a_1, \ldots, a_5$; $b_1, \ldots, b_{10}$ and $c_1, \ldots, c_5$.
Now 1 cannot be at a_4, a_5, b_7, b_8 or c_4 hence 1 must be at c_3.
Hence b_4 and b_5 are 2 and 5 in either order. Hence a_3 is 1 or 4.
But 1 cannot be at a_4 or b_7 hence 1 must be at a_3.
4 cannot be at b_3 thus 4 is at a_2.
Hence the number on the face with the question mark must be 4.

24. **B** A shaded triangle is congruent to an unshaded triangle (ASA).
Hence the area of the dashed square is equal to the area of the cross and both are 5.
Thus the side-length of the dashed square is $\sqrt{5}$.
Hence the sides of a shaded triangle are: $\frac{1}{2}$, 1 and $\frac{1}{2}\sqrt{5}$.
Now the perpendicular distance between the squares is equal to the altitude, h, of the shaded triangle. The area of such a triangle is $\frac{1}{2} \times \left(\frac{1}{2} \times 1\right) = \frac{1}{4}$ so that $\frac{1}{2} \times \left(\frac{1}{2}\sqrt{5} \times h\right) = \frac{1}{4}$ which gives $h = \frac{1}{\sqrt{5}}$.



Hence the length of the sides of the outer square are $\sqrt{5} + 2 \times \frac{1}{\sqrt{5}} = \frac{5}{\sqrt{5}} + \frac{2}{\sqrt{5}} = \frac{7}{\sqrt{5}}$.

Thus the area of the large square is $\left(\frac{7}{\sqrt{5}}\right)^2 = \frac{49}{5}$.

25. **A** The left-hand side of the equation can be written as

$$(a + 1)(b + 1)(c + 1)(d + 1) - 1.$$

Hence

$$(a + 1)(b + 1)(c + 1)(d + 1) = 2010.$$

Now expressing 2010 as a product of primes gives $2010 = 2 \times 3 \times 5 \times 67$ hence $a + b + c + d = 1 + 2 + 4 + 66 = 73$.