1.	**C**
2.	**B**
3.	**D**
4.	**C**
5.	**E**
6.	**E**
7.	**D**
8.	**A**
9.	**D**
10.	**D**
11.	**B**
12.	**C**
13.	**C**
14.	**D**
15.	**A**
16.	**A**
17.	**E**
18.	**C**
19.	**B**
20.	**B**
21.	**B**
22.	**A**
23.	**B**
24.	**E**
25.	**D**

UK SENIOR MATHEMATICAL CHALLENGE

Organised by the **United Kingdom Mathematics Trust**

SOLUTIONS

Keep these solutions secure until after the test on

THURSDAY 6 NOVEMBER 2008

This solutions pamphlet outlines a solution for each problem on this year's paper. We have tried to give the most straightforward approach, but the solutions presented here are not the only possible solutions. Occasionally we have added a 'Note' (in italics).

Please share these solutions with your students.

Much of the potential benefit of grappling with challenging mathematical problems depends on teachers making time for some kind of review, or follow-up, during which students may begin to see what they should have done, and how many problems they could have solved.

We hope that you and they agree that the first 15 problems could, in principle, have been solved by most candidates; if not, please let us know.

1. **C** $2 \times 2008 + 2008 \times 8 = 10 \times 2008 = 20080$.

2. **B** The cost per pound is £$\frac{255}{1250} \approx$ £$\frac{1}{5} = 20$ p.

3. **D** $\frac{1}{2^6} + \frac{1}{6^2} = \frac{3^2 + 2^4}{2^6 \times 3^2} = \frac{25}{2^6 \times 3^2} = \frac{5^2}{(2^3 \times 3)^2}$. Hence the answer is $\frac{5}{2^3 \times 3} = \frac{5}{24}$.

4. **C** From the units column we see that $S = 0$. Then the tens column shows that $R = 9$, the hundreds column that $Q = 1$, and the thousands that $P = 6$. So $P + Q + R + S = 16$.

5. **E** Since 1% of £400 = £4, the total VAT charged was £4 $\times$ 17.5 = £70, giving a total cost of £400 + £70 = £470. Therefore the minimum number of entries needed is 94.

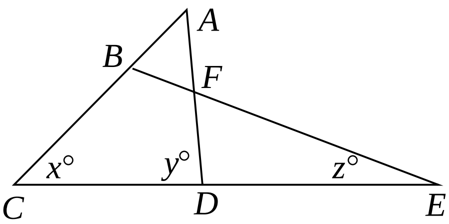
6. **E**

6		
4	5	
1	2	3

We number the squares to identify them. The only line of symmetry possible is the diagonal through 1 and 5. For a symmetric shading, if 4 is shaded, then so too must be 2; so either both are shaded or neither. Likewise 3 and 6 go together and provide 2 more choices. Whether 1 is shaded or not will not affect a symmetry, and this gives a further 2 choices; and the same applies to 5. Overall, therefore, there are $2^4 = 16$ choices. However, one of these is the choice to shade no squares, which is excluded by the question.

7. **D** In 1.8 miles there are 1.8×5280 feet $= 18 \times 528$ feet, while in 8 months there are roughly $8 \times 30 \times 24 \times 60$ minutes. Hence the time to 'run' one foot in minutes is roughly $\frac{10 \times 30 \times 20 \times 60}{20 \times 500} = 36$ minutes.

8. **A** In triangle ACD, $\angle CAD = (180 - x - y)°$.
As $AB = AF$, triangle ABF is isosceles hence $\angle ABF = \angle AFB = \frac{1}{2}(x + y)°$.
Thus $\angle DFE = \angle AFB = \frac{1}{2}(x + y)°$ (vertically opposite angles). Now in triangle DFE, $\angle FDE = (180 - y)°$. Hence
$z° = 180° - \angle DFE - \angle FDE = \frac{1}{2}(y - x)°$.



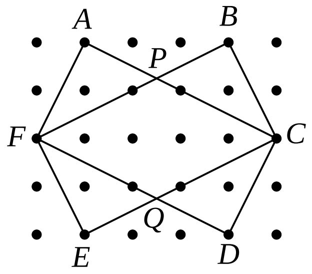
9. **D** A number is divisible by 9 if, and only if, the sum of its digits is divisible by 9. The given number is $N + 2$, where $N = 222\ldots220$ has 2007 2s. Since $2007 = 223 \times 9$, N is divisible by 9 and the required remainder is therefore 2.

10. **D** By inspection

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{4}; \quad \frac{3}{5} = \frac{1}{2} + \frac{1}{10}; \quad \frac{3}{6} = \frac{1}{3} + \frac{1}{6}; \quad \frac{3}{8} = \frac{1}{4} + \frac{1}{8}.$$

However $\frac{3}{7} \neq \frac{1}{m} + \frac{1}{n}$. [To see why, suppose that $\frac{3}{7} = \frac{1}{m} + \frac{1}{n}$ and note that $\frac{1}{m} > \frac{1}{n}$ or vice versa. We will suppose the former. Then $\frac{1}{m} \geqslant \frac{3}{14} > \frac{3}{15}$ and so $\frac{1}{m} > \frac{1}{5}$ and $m < 5$. Also $\frac{1}{m} < \frac{3}{7}$ and so $3m > 7$. Hence $m \geqslant 3$. So $m = 4$ or $m = 3$. However $\frac{3}{7} - \frac{1}{4} = \frac{5}{28}$ and $\frac{3}{7} - \frac{1}{3} = \frac{2}{21}$ neither of which has the form $\frac{1}{n}$.]

11. **B** Let the six points where lines meet on the dot lattice be A, B, C, D, E, F as shown and let the other two points of intersection be P (where AC and BF meet) and Q (where CE and DF meet).

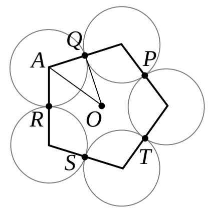


Triangles APB and CPF are similar with base lengths in the ratio 3:5. Hence triangle CPF has height $\frac{5}{8} \times 2 = \frac{5}{4}$ units and base length 5 units so that its area is $\frac{1}{2} \times \frac{5}{4} \times 5$ square units. Since the same is true of triangle CQF, the required area is $\frac{5}{4} \times 5 = 6\frac{1}{4}$ square units.

12. **C** There are 365 days in a normal year and 366 in a leap year. Apart from certain exceptions (none of which occurs in this period) a leap year occurs every 4 years. Now $365 = 7 \times 52 + 1$ and $366 = 7 \times 52 + 2$. Hence each date moves on by 5 days every 4 years. So in 60 years, it moves on 75 days. Since $75 = 7 \times 10 + 5$, that means it moves on to a Thursday.

13. **C** Since $1280 = 2^8 \times 5 = 2^8(2^0 + 2^2) = 2^8 + 2^{10}$, we may take m=8 and $n = 10$ (or vice versa) to get $m + n = 8 + 10 = 18$. It is easy to check that there are no other possibilities.

14. **D** The internal angle of a regular pentagon is 108°. Let A be the centre of a touching circle, as shown. Since OA bisects $\angle RAQ$, $\angle OAQ = 54°$. Also, triangle OAQ is right-angled at Q (radius perpendicular to tangent). Since $AQ = 1$, $OQ = \tan 54°$.



15. **A** The sequence proceeds as follows: 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1 … . The block 4, 2, 1 repeats *ad infinitum* starting after t_7. But $2008 - 7 = 2001$ and $2001 = 3 \times 667$. Hence t_{2008} is the third term in the 667th such block and is therefore 1.

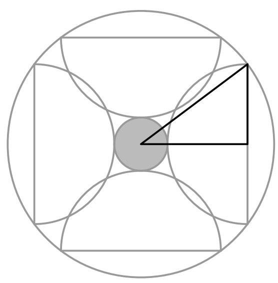
16. **A** Adding the three given equations gives $4(x + y + z) = 3000$. Therefore $x + y + z = 750$. So the mean is $\frac{750}{3} = 250$.

17. **E** Let 'X' be a single digit. If $2008 - 200X = 2 + 0 + 0 + X$ then $8 - X = 2 + X$ so $X = 3$. So Alice (being the younger) could have been born in 2003. Next if $2008 - 199X = 1 + 9 + 9 + X$ then $18 - X = 19 + X$, which is impossible. Similarly if $2008 - 198X = 1 + 9 + 8 + X$ then $28 - X = 18 + X$, so $X = 5$. Thus Alice or Andy could have been born in 1985. Finally if $2008 - 19YX = 1 + 9 + X + Y$ for some digit $Y \leqslant 7$, then $108 - YX = 10 + Y + X$. Hence $98 = YX + Y + X$ which is impossible, since $YX + Y + X$ is at most $79 + 7 + 9 = 95$. Hence there are no more possible dates and so Andy was born in 1985 and Alice in 2003.

18. **C** Since $XY^2 = 18$, $YZ^2 = 32$ and $XZ^2 = 50$, we have $XZ^2 = XY^2 + YZ^2$. Hence by the converse of Pythagoras' Theorem, $\angle XYZ = 90°$. Since the angle in a semi-circle is 90° the segment XZ is the diameter of the specified circle. Hence the radius is $\frac{1}{2}\sqrt{50}$ and the area of the circle is $\dfrac{50\pi}{4} = \dfrac{25\pi}{2}$.

19. **B** Let $199p + 1 = X^2$. Then $199p = X^2 - 1 = (X + 1)(X - 1)$. Note that 197 is prime. If p is also to be prime then **either** $X + 1 = 199$, in which case $X - 1 = 197$, **or** $X - 1 = 199$, in which case $X + 1 = 201$ (and $201 = 3 \times 67$ is not prime). Note that $X - 1 = 1, X + 1 = 199p$ is impossible. Hence $p = 197$ is the only possibility.

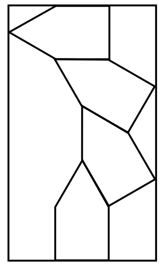
20. **B** Let r_1, r_2 and r_3 be the radii of the shaded circle, semicircles and outer circle respectively. A right-angled triangle can be formed with sides r_3, $(r_1 + r_2)$ and r_2.
Hence, by Pythagoras' Theorem, $r_3^2 = (r_1 + r_2)^2 + r_2^2$.
Now $\pi r_1^2 = 4$, hence $r_1 = 2/\sqrt{\pi}$. Likewise $r_2 = 6/\sqrt{\pi}$.
Hence $r_2 = 3r_1$ so that $r_3^2 = (r_1 + 3r_1)^2 + (3r_1)^2 = 25r_1^2$. Thus the required area is $25 \times 4 = 100$.



21. **B** Since 2008/1998 lies between 1 and 2, $a = 1$. Subtracting 1 and inverting gives $b + 1/(c + 1/d) = 1998/10 = 199 + 4/5$ so that $b = 199$. Then $1/(c + 1/d) = 4/5$ so that $c + 1/d = 5/4$ and this gives $c = 1$ and $d = 4$.
{*Note : This is an example of a continued fraction.*}

22. **A** Let r be the length of a side of the equilateral triangle.
Hence the width of the rectangle is $r\sin 60° + r + r\sin 60° = r(1 + 2\sin 60°) = r(1 + \sqrt{3})$ and its length is $3r + 2r\sin 60° = r(3 + \sqrt{3})$.
So the ratio of the length to the width is

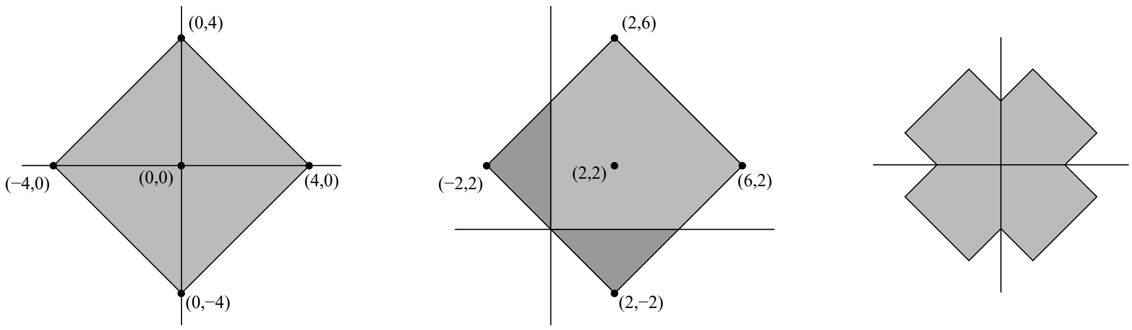
$$(3 + \sqrt{3}) : (1 + \sqrt{3}) = \sqrt{3}(1 + \sqrt{3}) : (1 + \sqrt{3}) = \sqrt{3} : 1.$$



23. **B** Let $X = x + 3$ and $Y = y - 3$. Then the given equation becomes $(X + Y)^2 = XY$. So $X^2 + XY + Y^2 = 0$. However X^2, Y^2 and $XY\ (= (X + Y)^2)$ are non-negative. Hence $X = Y = 0$; so $x = -3$ and $y = 3$ is the only solution.

24. **E** $1 + 3 + 5 + 7 + \ldots + (2n + 1) = (n + 1)^2$. The n in the three cases given is 12, $\frac{1}{2}(x - 1)$ and $\frac{1}{2}(y - 1)$. So, the triangle has sides of length $12 + 1$, $\frac{1}{2}(x - 1) + 1$ and $\frac{1}{2}(y - 1) + 1$. However the only right-angled triangle having sides of whole number length with hypotenuse 13 is the (5, 12, 13) triangle. So $x = 9$ and $y = 23$ (or vice versa). Hence $x + y = 32$.

25. **D** To work out the area of $\big||x| - 2\big| + \big||y| - 2\big| \leqslant 4$, we first consider the region $|x| + |y| \leqslant 4$ which is shown in (a). This region is then translated to give $|x - 2| + |y - 2| \leqslant 4$ as shown in (b).
By properties of the modulus, if the point (x, y) lies in the polygon, then so do $(x, -y)$, $(-x, y)$ and $(-x, -y)$. Thus $\big||x| - 2\big| + \big||y| - 2\big| \leqslant 4$ can be obtained from (b) by reflecting in the axes and the origin, as shown in (c).



(a) (b) (c)

Hence the required area is 4 times the area in the first quadrant. From (b), the required area in the first quadrant is the area of a square of side $4\sqrt{2}$ minus two triangles (cut off by the axes) which, combined, make up a square of side $2\sqrt{2}$. So the area in the first quadrant is $(4\sqrt{2})^2 - (2\sqrt{2})^2 = 32 - 8 = 24$.
Hence the area of the polygon is $4 \times 24 = 96$ square units.