1.	C
2.	A
3.	B
4.	C
5.	E
6.	A
7.	D
8.	E
9.	E
10.	B
11.	D
12.	E
13.	B
14.	B
15.	D
16.	A
17.	C
18.	A
19.	D
20.	C
21.	B
22.	D
23.	E
24.	D
25.	D

UK SENIOR MATHEMATICAL CHALLENGE

Organised by the **United Kingdom Mathematics Trust**

SOLUTIONS

Keep these solutions secure until after the test on

THURSDAY 8 NOVEMBER 2007

This solutions pamphlet outlines a solution for each problem on this year's paper. We have tried to give the most straightforward approach, but the solutions presented here are not the only possible solutions. Occasionally we have added a 'Note' (in italics).

Please share these solutions with your students.

Much of the potential benefit of grappling with challenging mathematical problems depends on teachers making time for some kind of review, or follow-up, during which students may begin to see what they should have done, and how many problems they could have solved.

We hope that you and they agree that the first 15 problems could, in principle, have been solved by most candidates; if not, please let us know.

1. **C** $\frac{2007}{9} + \frac{7002}{9} = \frac{9009}{9} = 1001.$

2. **A** If Sam's birthday falls before 9 November, then the fact that she is aged 30 on 8 November means that she was born in 1977. However, if her birthday falls on 9 November or later then her 31st birthday will fall in 2007, which means that she was born in 1976.

3. **B** In general, $(n-1) \times (n+1) - n^2 = n^2 - 1 - n^2 = -1$. This applies with $n = 2007$.

4. **C** $\angle WPQ = 120°$ (interior angle of a regular hexagon), so $\angle WPS = (360 - 120 - 90)° = 150°$. Now $PW = PQ$ (sides of a regular hexagon) and $PS = PQ$ (sides of a square) so $PW = PS$. Therefore triangle PSW is isosceles and $\angle PSW = (180 - 150)° \div 2 = 15°$.

5. **E** $4^4 = (2^2)^4 = 2^8$; $8^{8/3} = (2^3)^{8/3} = 2^8$; $16^2 = (2^4)^2 = 2^8$. However, $32^{6/5} = (2^5)^{6/5} = 2^6$.

6. **A** Let the number of five-pence coins be x. Then $5x + 2(50 - x) = 181$, that is $3x = 81$, that is $x = 27$. So there are 27 five-pence coins and 23 two-pence coins.

7. **D** There are 1003 whole numbers between 1 and 2007 which are divisible by 2. Those which are also divisible by 7 are the multiples of 14, namely 14, 28, 42, …, 2002. There are 143 of these, so the required number is $1003 - 143 = 860$.

8. **E** The distance travelled to Birmingham by the train was 300 km. The time taken to travel this distance at an average speed of 90 km/hr is $\frac{300}{90}$ hr $= 3\frac{1}{3}$ hr $=$ 3 hr 20 min. So the train was waiting for 20 minutes.

9. **E** Let the original cost price and original selling price of the dress be C and S respectively. Then $0.8 \times S = 1.04 \times C$. So $S = \frac{1.04}{0.8} \times C = 1.3 \times C$.
Therefore the shopkeeper would have made a profit of 30% by selling the dress at its original price.

10. **B** The volume of water which fell at Sprinkling Tarn in 1954 is approximately equal to $(25\,000 \times 6)$ m^3, that is 150 000 m^3. Now 1 m^3 $= 10^6$ cm^3 $= 10^6$ ml $=$ 1000 litres. So approximately 150 million litres of water fell on Sprinkling Tarn in 1954.

11. **D** Each of the original faces of the cube now has area $4 \times 4 - 2 \times 2$, that is 12. In addition, the drilling of the holes has created 24 rectangles, each measuring 2×1. So the required area is $6 \times 12 + 24 \times 2 = 120$.

12. **E** Let N be the two-digit number 'ab', that is $N = 10a + b$. So the sum of N and its 'reverse' is $10a + b + 10b + a = 11a + 11b = 11(a + b)$. As 11 is prime and a and b are both single digits, $11(a + b)$ is a square if, and only if, $a + b = 11$. So the possible values of N are 29, 38, 47, 56, 65, 74, 83, 92.

13. **B** The exact number of seconds in six complete weeks is $6 \times 7 \times 24 \times 60 \times 60 =$ $6 \times 7 \times (3 \times 8) \times (2 \times 5 \times 6) \times (3 \times 4 \times 5) = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 = 10!$.

14. **B** The shaded area is $\frac{x}{360}(\pi \times 4^2 - \pi \times 1^2) = \frac{15\pi x}{360} = \frac{\pi x}{24}$. So $\frac{\pi x}{24} = \frac{\pi \times 4^2}{6}$; thus $x = 64$.

15. **D** In the given diagram, there are four hexagons congruent to the hexagon in Figure (i), four hexagons congruent to the hexagon in Figure (ii) and eight hexagons congruent to the hexagon in Figure (iii).

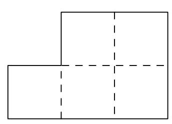


Figure (i)

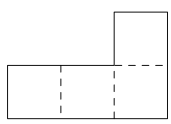


Figure (ii)

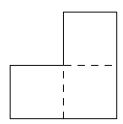
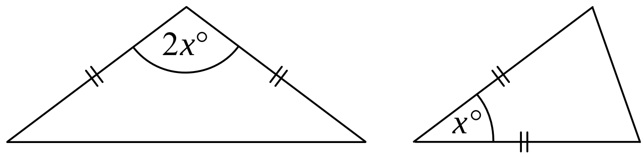


Figure (iii)

16. **A** The smallest number of possible prime divisors of 457 that Damien needs to check is the number of prime numbers less than or equal to the square root of 457. Since $21^2 < 457 < 22^2$, he needs to check only primes less than 22. These primes are 2, 3, 5, 7, 11, 13, 17 and 19.

17. **C** Let the equal sides have length k. The height of the triangle on the left is $k\cos x°$ and its base is $2k\sin x°$, so its area is $k^2\sin x°\cos x°$. The height of the triangle on the right is $k\sin x°$ and its base is k, so its area is $\frac{1}{2}k^2\sin x°$. Hence $\cos x° = \frac{1}{2}$ and so $x = 60$.

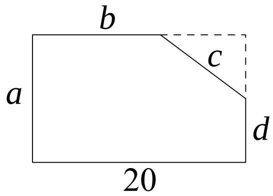


(*Alternatively, the formula* $\Delta = \frac{1}{2}ab\sin C$ *can be used to show that* $\sin x° = \sin 2x°$; *hence* $x + 2x = 180$.)

18. **A** There are 9 years of the form 123n as n may be any digit other than 4. Similarly, there are 9 years each of the forms 234n, 345n, 456n, 567n and 678n, but 10 years of the form 789n as, in this case, n may be *any* digit. There are also 9 years of the form n012 and 9 of the form n123, as in both cases n may be any digit other than 0. However, there are 8 years of the form n234 as in this case n cannot be 0 or 1. Similarly, there are 8 years each of the forms n345, n456, n567, n678 and n789.
So the total numbers of years is $1 \times 10 + 8 \times 9 + 6 \times 8 = 130$.

19. **D** By the Alternate Segment Theorem $\angle QUS = 55°$. Tangents to a circle from an exterior point are equal, so $QU = QS$ and hence $\angle QSU = \angle QUS = 55°$. So $\angle PQR = 180° - 2 \times 55° = 70°$.

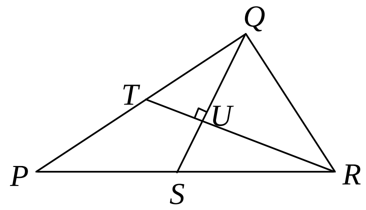
20. **C** The diagram shows the original rectangle with the corner cut from it to form a pentagon. It may be deduced that the length of the original rectangle is 20 and that a, b, c, d are 8, 10, 13, 15 in some order.



By Pythagoras' Theorem $c^2 = (20 - b)^2 + (a - d)^2$. So c cannot be 8 as there is no right-angled triangle having integer sides and hypotenuse 8. If $c = 10$, then $(20 - b)$ and $(a - d)$ are 6 and 8 in some order, but this is not possible using values of 8, 13 and 15. Similarly, if $c = 15$, then $(20 - b)$ and $(a - d)$ are 9 and 12 in some order, but this is not possible using values of 8, 10 and 13. However, if $c = 13$, then $(20 - b)$ and $(a - d)$ are 5 and 12 in some order, which is true if and only if $a = 15$, $b = 8$, $d = 10$.
So the area of the pentagon is $20 \times 15 - \frac{1}{2} \times 5 \times 12 = 270$.

21. **B** In this solution, the notation p / q / r / s / … represents p beads of one colour, followed by q beads of the other colour, followed by r beads of the first colour, followed by s beads of the second colour etc.
Since the colours alternate, there must be an even number of these sections of beads. If there are just two sections, then the necklace is 4/4 and there is only one such necklace. If there are four, then each colour is split either 2, 2 or 3, 1. So the possibilities are 2/3/2/1 (which can occur in two ways, with the 3 being one colour or the other) or 2/2/2/2 (which can occur in one way) or 3/3/1/1(also one way). Note that 3/2/1/2 appears to be another possibility, but is the same as 2/3/2/1 rotated.
If there are six sections, then each colour must be split into 2, 1, 1 and the possibilities are 2/2/1/1/1/1 (one way)or 2/1/1/2/1/1 (one way). Finally, if there are eight, then the only possible necklace is 1/1/1/1/1/1/1/1. In total that gives 8 necklaces.

22. D Let U be the point of intersection of QS and RT. As QS and RT are medians of the triangle, they intersect at a point which divides each in the ratio 2:1, so $QU = \frac{2}{3} \times 8 = \frac{16}{3}$. Therefore the area of triangle $QTR = \frac{1}{2} \times RT \times QU = \frac{1}{2} \times 12 \times \frac{16}{3} = 32$.

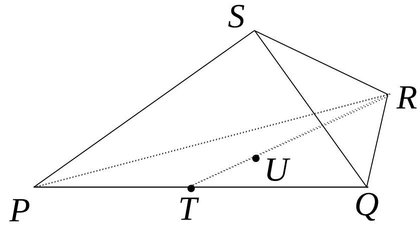


As a median of a triangle divides it into two triangles of equal area, the area of triangle PTR is equal to the area of triangle QTR, so the area of triangle PQR is 64.

23. E Let the lengths of the sides of the cuboid, in cm, be a, b and c. So $4(a + b + c) = x$. Also, by Pythagoras' Theorem $a^2 + b^2 + c^2 = y^2$. Now the total surface area of the cuboid is

$$2ab + 2bc + 2ca = (a + b + c)^2 - (a^2 + b^2 + c^2) = \left(\frac{x}{4}\right)^2 - y^2 = \frac{x^2 - 16y^2}{16}.$$

24. D The diameter of the sphere is $l - 2h$ where l is the length of a space diagonal of the cube and h is the perpendicular height of one of the tetrahedral corners when its base is an equilateral triangle.



The diagram shows such a tetrahedron: S is a corner of the cube; the base of the tetrahedron, which is considered to lie in a horizontal plane, is an equilateral triangle, PQR, of side $\sqrt{2}$ units; T is the midpoint of PQ. Also U is the centroid of triangle PQR, so $RU : UT = 2 : 1$. As U is vertically below S, the perpendicular height of the tetrahedron is SU.

As RTP is a right angle, $RT^2 = RP^2 - TP^2 = (\sqrt{2})^2 - \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{3}{2}$. Also, $RU = \frac{2}{3}RT$, so $RU^2 = \frac{4}{9}RT^2 = \frac{4}{9} \times \frac{3}{2} = \frac{2}{3}$.
So $SU^2 = SR^2 - RU^2 = 1 - \frac{2}{3} = \frac{1}{3}$. Therefore $h = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}$.
Now $l^2 = 2^2 + 2^2 + 2^2 = 12$, so $l = \sqrt{12} = 2\sqrt{3}$. Therefore the diameter of the sphere is $2\sqrt{3} - 2 \times \frac{\sqrt{3}}{3} = \frac{4\sqrt{3}}{3}$.

25. D As the line $y = x$ is an axis of symmetry of the curve, if the point (a, b) lies on the curve, so too does the point (b, a). Hence the equation of the curve may also be written as $x = \frac{py + q}{ry + s}$.
Therefore, substituting for x in the original equation:

$$y = \frac{p\left(\frac{py + q}{ry + s}\right) + q}{r\left(\frac{py + q}{ry + s}\right) + s} = \frac{p(py + q) + q(ry + s)}{r(py + q) + s(ry + s)}.$$

Therefore $$y\big(r(py + q) + s(ry + s)\big) = p(py + q) + q(ry + s),$$

that is $y^2 r(p + s) + y\left(qr + s^2 - p^2 - qr\right) - pq - qs = 0$,

that is $$(p + s)\left(y^2 r + y(s - p) - q\right) = 0.$$

Since r is non-zero, the expression in the second bracket is non-zero for all but at most two values of y. Hence $p + s = 0$.