Junior Mathematical Challenge

Organised by the United Kingdom Mathematics Trust

supported by Institute and Faculty of Actuaries

Solutions and investigations

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some exercises for further investigation. We welcome comments on these solutions. Please send them to enquiry@ukmt.org.uk.

The Junior Mathematical Challenge (JMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the JMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. So for each question we have included a complete solution which does not use the fact that one of the given alternatives is correct. Thus we have aimed to give full solutions with all steps explained. We therefore hope that these solutions can be used as a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the Junior Mathematical Olympiad and similar competitions).

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
B	A	E	C	A	C	A	A	E	D	C	B	C	D	E	B	C	D	B	E	A	E	B	D	D

1. Which of the following is closest to zero?

A $6+5+4$ B $6+5-4$ C $6+5\times4$ D $6-5\times4$ E $6\times5\div4$

SOLUTION **B**

We work out the value of each expression in turn.

A	$6+5+4 = 11+4 = 15.$
B	$6+5-4 = 11-4 = 7.$
C	$6+5\times4 = 6+20 = 26.$
D	$6-5\times4 = 6-20 = -14.$
E	$6\times5\div4 = 6\times1.25 = 7.5.$

We see that expression B has the value that is closest to zero.

NOTE

Note that in evaluating these expressions we use the standard convention that divisions and multiplications are carried out before additions and subtractions.

For example, we evaluate expression D as $6-(5\times4)$ and *not* as $(6-5)\times4$.

Check your calculator. When you press the keys

[6] [−] [5] [×] [4] [=]

in this order, the answer should be -14. If your calculator gives you the answer 4, or anything else, it is time to buy a new calculator.

FOR INVESTIGATION

1.1 Work out the values of the following expressions.

(a) $1\times2+3\times4+5\times6.$

(b) $1\times(2+3)\times(4+5)\times6.$

(c) $1\times2-3\times4+5\times6.$

(d) $(-1-2+(3+4)\times5)\times(6-7+8)\times9.$

2. What number is twenty-one less than sixty thousand?

A 59 979 B 59 981 C 57 900 D 40 001 E 39 000

SOLUTION **A**

METHOD 1

The direct method is to do the subtraction sum, as follows.

$$\begin{array}{r} 6\ 0\ 0\ 0\ 0 \\ -\quad 2\ 1 \\ \hline 5\ 9\ 9\ 7\ 9 \\ \hline \end{array}$$

So the answer is 59 979.

METHOD 2

Without doing the full subtraction we can see that the units digit of $60\,000 - 21$ is 9. In the context of the JMC you are entitled to assume that one of the given options is correct. As 59 979 is the only option with units digit 9, we deduce that this is the correct answer.

3. One lap of a standard running track is 400 m.

How many laps does each athlete run in a 5000 m race?

A 4 B 5 C 8 D 10 E $12\frac{1}{2}$

SOLUTION **E**

Since each lap has length 400 m and each athlete runs 5000 m, the number of laps that each athlete runs is

$$\frac{5000}{400} = \frac{50}{4} = 12\tfrac{1}{2}.$$

4. In January 1859, an eight-year-old boy dropped a newly-hatched eel into a well in Sweden (apparently in order to keep the water free of insects). The eel, named Åle, finally died in August 2014.

How many years old was Åle when it died?

A 135 B 145 C 155 D 165 E 175

Solution **C**

To find Åle's age we need to work out $2014 - 1859$.

Method 1

We can do the subtraction as follows:

$$\begin{array}{r} 2\ 0\ 1\ 4 \\ -\ \ 1\ 8\ 5\ 9 \\ \hline 1\ 5\ 5 \\ \hline \end{array}$$

So Åle's age when it died was 155.

Method 2

Alternatively, we can see that from 1859 to 1900 is 41 years, from 1900 to 2000 is 100 years, and from 2000 to 2014 is 14 years. So Åle's age in years is $41 + 100 + 14$, that is, 155.

For investigation

4.1 In which year was Åle's age 123 years?

4.2 In which year was Åle ten times as old as you are now?

5. What is the value of $\frac{1}{25} + 0.25$?

A 0.29 B 0.3 C 0.35 D 0.50 E 0.65

Solution **A**

We first convert the fraction into a decimal. We multiply both the numerator (the top) and the denominator (the bottom) by 4. This gives $\frac{1}{25} = \frac{4}{100}$. Therefore, expressed as a decimal, $\frac{1}{25}$ is 0.04. Hence

$$\frac{1}{25} + 0.25 = 0.04 + 0.25 = 0.29.$$

6. Gill is now 28 years old and is a teacher of Mathematics at a school which has 600 pupils. There are 30 more girls than boys at the school.

How many girls are at Gill's school?

A 270 B 300 C 315 D 330 E 345

Solution **C**

Method 1

Suppose there were equal numbers of girls and boys at the school. Then there would be 300 girls and 300 boys. Each time we add one more girl and take away one boy, the difference between the number of girls and the number of boys goes up by 2, while the total number of pupils remains 600. So to have 30 more girls than boys, we would need to add 15 more girls and take away 15 boys. So a school with 600 pupils and 30 more girls than boys has 315 girls.

Method 2

Suppose that the school has g girls. Then, as there are 600 pupils altogether, there are $600 - g$ boys. Since there are 30 more girls than boys,

$$g - (600 - g) = 30.$$

This gives

$$g - 600 + g = 30,$$

that is,

$$2g - 600 = 30,$$

and hence

$$2g = 630.$$

Therefore

$$g = 315.$$

Hence, there are 315 girls at Gill's school.

7. A distance of 8 km is approximately 5 miles.

Which of the following is closest to 1.2 km?

A 0.75 miles B 1 mile C 1.2 miles D 1.6 miles E 1.9 miles

Solution **A**

Method 1

Since 8 km is approximately 5 miles, 1 km is approximately $\frac{5}{8}$ miles. Hence 1.2 km is approximately

$$1.2 \times \frac{5}{8} \text{ miles.}$$

We have

$$1.2 \times \frac{5}{8} = \frac{1.2 \times 5}{8} = \frac{6}{8} = 0.75.$$

Therefore 1.2 km is approximately 0.75 miles.

Method 2

Since 8 km is approximately 5 miles, 1 mile is approximately $\frac{8}{5}$ km, that is, 1.6 km. It follows that 1.2 km is significantly less than 1 mile. So 0.75 miles is the correct option.

More precisely, it follows that 1.2 km is approximately $\frac{1.2}{1.6}$ miles. Since

$$\frac{1.2}{1.6} = \frac{12}{16} = \frac{3}{4} = 0.75,$$

it follows that 1.2 km is approximately 0.75 miles.

For investigation

7.1 It is 393 miles from London to Edinburgh by train. Approximately how many kilometres is this distance?

7.2 The length of the marathon, as run at the Olympic games, has been standardized at 42.195 km since 1921. This distance is based on the length of the marathon at the 1908 Olympic Games in London, when the race started at Windsor Castle and ended at the White City Stadium in London.

(a) What is the distance 42.195 km, to the nearest mile?

(b) 1 km is exactly 0.6213712 miles. There are 1760 yards in 1 mile. What is the length of the Olympic marathon in miles and yards, to the nearest yard?

[*Calculators allowed*!]

8. What is the value of

$$\frac{2+4+6+8+10+12+14+16+18+20}{1+2+3+4+5+6+7+8+9+10}?$$

A 2 B 10 C 20 D 40 E 1024

SOLUTION **A**

Each number in the numerator is a multiple of 2. If we take out this common factor, we obtain

$$2+4+6+8+10+12+14+16+18+20 = 2\times(1+2+3+4+5+6+7+8+9+10).$$

It follows that

$$\begin{aligned}\frac{2+4+6+8+10+12+14+16+18+20}{1+2+3+4+5+6+7+8+9+10} &= \frac{2\times(1+2+3+4+5+6+7+8+9+10)}{1+2+3+4+5+6+7+8+9+10}\\ &= 2.\end{aligned}$$

FOR INVESTIGATION

8.1 What is the value of

$$\frac{2\times4\times6\times8\times10\times12\times14\times16\times18\times20}{1\times2\times3\times4\times5\times6\times7\times8\times9\times10}?$$

9. One of the three symbols $+$, $-$, $\times$ is inserted somewhere between the digits of 2016 to give a new number. For example, $20-16$ gives 4.

How many of the following four numbers can be obtained in this way?

36	195	207	320

A 0 B 1 C 2 D 3 E 4

SOLUTION **E**

We see that

$$\begin{aligned}20+16 &= 36.\\ 201-6 &= 195.\\ 201+6 &= 207.\\ 20\times16 &= 320.\end{aligned}$$

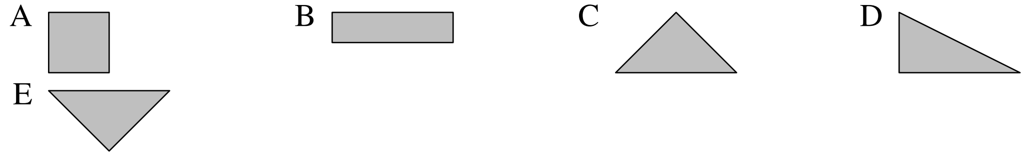
So we can obtain all four of the given numbers.

FOR INVESTIGATION

9.1 How many other numbers can you obtain by inserting one of the three symbols $+$, $-$, $\times$ somewhere between the digits of 2016?

10. A square is folded exactly in half and then in half again.

Which of the following could not be the resulting shape?



Solution **D**

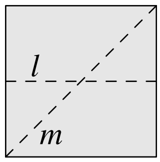


Figure 1

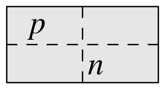


Figure 2

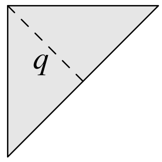


Figure 3

To fold a figure exactly in half we need to fold it along a line of symmetry.

A square has four lines of symmetry, but these are of just two types. There are two lines of symmetry joining the midpoints of opposite edges, and two lines of symmetry joining opposite vertices (see figure 1).

When a square is folded along a line of symmetry joining the midpoints of edges, such as l, the resulting shape is a rectangle whose longer sides are twice as long as the shorter sides (see figure 2). When it is folded along a line of symmetry joining opposite vertices, such as m, the resulting shape is an isosceles right-angled triangle (see figure 3).

The rectangle shown in figure 2 has two lines of symmetry, n and p, joining midpoints of opposite edges. When the rectangle is folded along n, the resulting shape is a square with half the side length of the original square. This is the shape of option A.

When the rectangle is folded along p, the resulting shape is a rectangle whose longer sides are four times as long as the shorter sides. This is the shape of option B.

The isosceles right-angled triangle shown in figure 3 has one line of symmetry, labelled q. When the triangle is folded along this line the resulting shape is a smaller isosceles right-angled triangle. This is the shape of both options C and E.

We therefore see that the shapes of options A, B, C and E can be achieved. Since we have covered all the possible ways of folding the square in half and in half again, it follows that the shape that could not be achieved is D.

For investigation

10.1 Which shapes can be obtained by starting with a square and folding along a line of symmetry three times in succession?

10.2 Which shapes can be obtained by starting with a square and folding along a line of symmetry any finite number of times?

11. Which of the following statements is false?

A 12 is a multiple of 2
B 123 is a multiple of 3
C 1234 is a multiple of 4
D 12 345 is a multiple of 5
E 123 456 is a multiple of 6

SOLUTION **C**

METHOD 1

One method is to do the following divisions, to find out in which case the answer is not an integer.

A $12 \div 2 = 6$.
B $123 \div 3 = 41$.
C $1234 \div 4 = 308.5$.
D $12\,345 \div 5 = 2469$.
E $123\,456 \div 6 = 20576$.

From these calculations we see that 1234 is not a multiple of 4. So the statement that is false is C.

METHOD 2

We have the following tests for whether a positive integer is a multiple of n, for small values of n.

n	Test for being a multiple of n
2	The units digit is even.
3	The sum of the digits is a multiple of 3.
4	The tens and units digits taken together make an integer which is a multiple of 4.
5	The units digit is 0 or 5.
6	A multiple of 2 and a multiple of 3.

We apply these tests to the statements given in the options.

A The units digits of 12 is 2, which is even. So 12 is a multiple of 2.
B The sum of the digits of 123 is $1 + 2 + 3 = 6$. This is a multiple of 3. So 123 is a multiple of by 3.
C The tens and units digits make 34. This is not a multiple of 4. So 1234 is *not* a multiple of 4.
D The units digit is 5. So 12 345 is a multiple of 5.
E 123 456 has units digit 6 and the sum of its digits 21. So it is a multiple of 2 and 3, and hence of 6.

Therefore it is statement C that is false.

FOR INVESTIGATION

11.1 Explain why the tests for being a multiple of n, for $n = 2, 3, 4, 5$ and 6, work.

12. The musical *Rent* contains a song that starts "Five hundred and twenty five thousand six hundred minutes".

Which of the following is closest to this length of time?

A a week B a year C a decade D a century E a millennium

Solution **B**

The song refers to 525 600 minutes. Since there are 60 minutes in an hour, 525 600 minutes is

$$\frac{525600}{60} \text{ hours, that is, 8760 hours.}$$

Since there are 24 hours in one day, 8760 hours is

$$\frac{8760}{24} \text{ days, that is, 365 days.}$$

It follows that of the given options *a year* is closest to the length of time corresponding to the number of seconds mentioned in the song.

Note

A calendar year has 365 days in a normal year, but 366 days in a leap year such as the current year.

Astronomers use several different definitions of a year which differ a little from each other in length. One common definition is the *sidereal year* which is the time it takes the earth to complete one revolution about the sun, relative to the distant stars. The sidereal year has length 365.256 363 004 days.

Because this is not an exact number of days, a calendar with the same number of days in each year would not be synchronised with the position of the earth relative to the sun and so it would get out of step with the seasons.

Different cultures have solved this problem in different ways. In the most commonly used calendar, introduced by Pope Gregory XIII in 1582, the rules for the length of a year are:

> A year is a leap year if its number is a multiple of 4, except that if the number is a multiple of 100 but not of 400, then it is not a leap year. A standard year has 365 days. A leap year has 366 days.

For example, 2016 is a multiple of 4. Therefore the current year is a leap year. 2017 is not a multiple of 4, so next year will not be a leap year. 2000 is a multiple of 100 and also of 400. So the year 2000 was a leap year. However, 2100 is divisible by 100 by not by 400. So the year 2100 will not be a leap year.

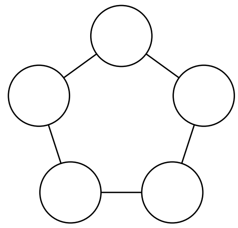
For investigation

12.1 According to the Gregorian calendar, how many leap years are there in a period of 400 years?

12.2 What is the average length, in days, of a year in the Gregorian calendar?

13. The diagram shows five circles placed at the corners of a pentagon. The numbers 1, 2, 3, 4, 5 are placed in the circles shown, one in each, so that the numbers in adjacent circles always differ by more than 1.

What is the sum of the numbers in the two circles adjacent to the circle which contains the number 5?



A 3 B 4 C 5 D 6 E 7

Solution **C**

The number 3 cannot be placed next to either 2 or 4. So it will be between 1 and 5. Therefore 5 is next to 3, and not next to 1. Also, 5 cannot be next to 4. Therefore 5 is next to 2.

Therefore the sum of the numbers next to 5 is 3 + 2, that is, 5. [In the JMC we could stop here. However for a complete answer we need to do Exercise 13.1.]

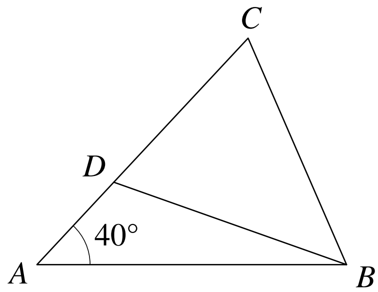
For investigation

13.1 Show that there is a way to place the numbers 1, 2, 3, 4, 5 in the circles so that the numbers in adjacent circles always differ by more than 1.

14. In the diagram, $AB = AC$ and D is a point on AC such that $BD = BC$. Angle BAC is 40°.

What is angle ABD ?

A 15° B 20° C 25° D 30° E 35°



Solution **D**

Since $AB = AC$, the triangle ABC is isosceles. Therefore $\angle ACB = \angle CBA$. Since the angles in a triangle add up to 180°, we have $\angle BAC + \angle ACB + \angle CBA = 180°$. Hence $40° + 2\angle ACB = 180°$.

It follows that $2\angle ACB = 180° - 40° = 140°$. Therefore $\angle ACB = 70°$.

Since $BD = BC$, the triangle BCD is isosceles. Therefore $\angle BDC = \angle DCB$. Since $\angle DCB$ is the same as $\angle ACB$, it follows that $\angle BDC = \angle ACB = 70°$.

By the External Angle Theorem, $\angle BDC = \angle BAD + \angle ABD$. That is, $70° = 40° + \angle ABD$. It follows that $\angle ABD = 30°$.

For investigation

14.1 If you are not familiar with the *External Angle Theorem*, find out what it says and how it is proved.

14.2 This problem may be solved without using the External Angle Theorem. Can you see how? [Nonetheless, do remember the External Angle Theorem - it is very useful.]

15. How many of these four expressions are perfect squares?

$1^3 + 2^3$	$1^3 + 2^3 + 3^3$	$1^3 + 2^3 + 3^3 + 4^3$	$1^3 + 2^3 + 3^3 + 4^3 + 5^3$

A 0 B 1 C 2 D 3 E 4

SOLUTION **E**

We find the value of each expression to see which of them are squares. After we have found the value of the one expression, we can find the value of the next expression by adding the extra term to the sum we have already calculated. In this way, we obtain

$$1^3 + 2^3 = 1 + 8 = 9 = 3^2.$$
$$1^3 + 2^3 + 3^3 = 9 + 3^3 = 9 + 27 = 36 = 6^2.$$
$$1^3 + 2^3 + 3^3 + 4^3 = 36 + 4^3 = 36 + 64 = 100 = 10^2.$$
$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 100 + 5^3 = 100 + 125 = 225 = 15^2.$$

We now see that all four of the expressions are perfect squares.

NOTE

The first cube 1^3 is, of course, also a square since $1^3 = 1^2$. We see from the above calculations that the sum of the first two cubes is square, and so also are the sums of the first three cubes, the first four cubes and the first five cubes. Just from these five examples we *cannot* deduce that the sum of the first n cubes is always a square, but it looks as though it might be true.

You should not be satisfied until you have found either a *proof* that this is always true, or a *counterexample* to show that it is false for at least one positive integer n.

FOR INVESTIGATION

15.1 If we include the case $1^3 = 1^2$, we see that the sums of the first n cubes, for $n = 1, 2, 3, 4$ and 5 are 1^2, 3^2, 6^2, 10^2 and 15^2. Can you spot a possible pattern?

15.2 Assuming that the pattern you have spotted in 15.1 is correct, write down a formula for the sum of the first n cubes.

15.3 Try to either find a proof that the formula that you wrote down in 15.2 is correct, or give an example of a positive integer n for which it is not true. You might look in a book, or ask your teacher, or read the note below.

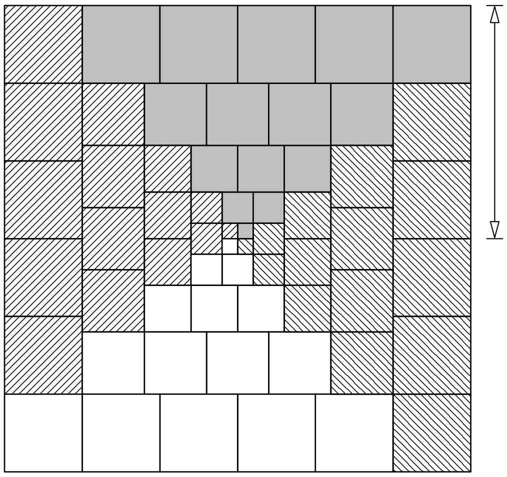
NOTE

From the numerical evidence, it looks as though the sum of the first n cubes equals the square of the sum of the first n positive integers. For example,

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 15^2 = (1 + 2 + 3 + 4 + 5)^2.$$

It turns out that this is true for all positive integers n. There are several different ways to prove this. The following proof, using a diagram and no symbols, is in many ways the most attractive.

Look at the square in the following diagram.



The square is made up of four congruent polygons, one white, one grey and two hatched.

Each of these polygons is made up of one square of size 1×1, two squares of size 2×2, three squares of size 3×3, four squares of size 4×4, and five squares of size 5×5. Therefore the area of each polygon is

$$1 \times (1 \times 1) + 2 \times (2 \times 2) + 3 \times (3 \times 3) + 4 \times (4 \times 4) + 5 \times (5 \times 5) = 1^3 + 2^3 + 3^3 + 4^3 + 5^3.$$

Since the square is made up four of these polygons, its area is

$$4 \times (1^3 + 2^3 + 3^3 + 4^3 + 5^3).$$

We also see that the height of the grey polygon is $1 + 2 + 3 + 4 + 5$. Therefore the height of the whole square is $2 \times (1 + 2 + 3 + 4 + 5)$. This is also the width of the square. Therefore the area of the square is

$$(2 \times (1 + 2 + 3 + 4 + 5))^2,$$

which is equivalent to

$$4 \times (1 + 2 + 3 + 4 + 5)^2.$$

Since these two different expressions for the area of the square are equal, it follows that

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = (1 + 2 + 3 + 4 + 5)^2.$$

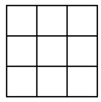
Clearly, this argument can be used to show that for each positive integer n, the sum of the first n cubes equals the square of the sum of the first n integers.

The visual proof we have given comes from an article *A Geometric Proof of a Famous Identity*, by Solomon Golomb, in the *Mathematical Gazette* (Volume 49, No. 368, May 1965, pages 198 - 200). In this article Golomb attributes the diagram on which the one given here is based to Warren Lushbaugh.

You may already know that the formula for the sum of the first n positive integers is $\frac{1}{2}n(n + 1)$. [If not, find a proof that this is the correct formula.] It follows that the sum of the first n positive cubes is given by the formula

$$\left(\tfrac{1}{2}n(n + 1)\right)^2, \text{ or, equivalently, } \tfrac{1}{4}n^2(n + 1)^2.$$

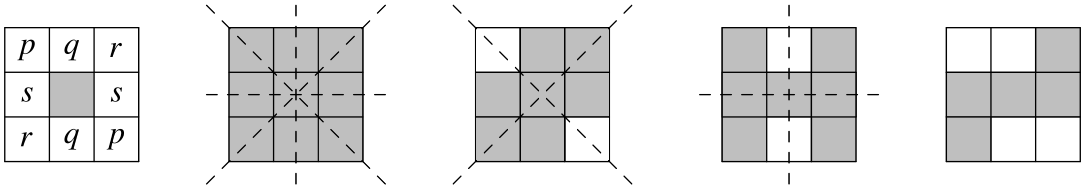
16. Each of the nine small squares in this grid can be coloured completely black or completely white.



What is the largest number of squares that can be coloured black so that the design created has rotational symmetry of order 2, but no lines of symmetry?

A 4 B 5 C 6 D 7 E 8

Solution **B**



We note first that the colour of the centre square does not affect the symmetries of the design. Since we are looking for the largest number of squares that can be coloured black, we need only consider colourings in which this square is black.

To have a colouring which has rotational symmetry of order 2, each pair of opposite squares, that is, the pairs labelled p, q, r and s in the diagram on the left, must be the same colour.

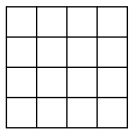
The second diagram above shows that, if all four of these pairs are black, then the design has four lines of symmetry. The third and fourth diagrams show that, if one of these pairs is coloured white and the other three pairs are black, then the design has two lines of symmetry.

The diagram on the right shows, however, that we can create a design with no lines of symmetry by colouring the appropriate two pairs white and the other two pairs black. It can be seen that the design in this diagram has rotational symmetry of order 2.

This shows that the largest number of squares that can be coloured black to make a design with rotational symmetry of order 2, but no lines of symmetry, is 5.

For investigation

16.1 Consider now a 4×4 grid with 16 small squares.



What is the largest number of squares that can be coloured black, with the rest coloured white, so that the design has rotational symmetry of order 2, but no lines of symmetry?

16.2 Suppose we have an $n \times n$ grid made up of n^2 small squares.

What is the largest number of squares that can be coloured black, with the rest coloured white, so that the design has rotational symmetry of order 2, but no lines of symmetry?

17. In a group of 48 children, the ratio of boys to girls is 3 : 5.

How many boys must join the group to make the ratio of boys to girls 5 : 3?

A 48 B 40 C 32 D 24 E 8

SOLUTION **C**

Since the ratio of boys to girls is 3 : 5, the proportion of boys in the group is $\frac{3}{8}$. Therefore the number of boys is

$$\frac{3}{8} \times 48 = 18.$$

It follows that the number of girls in the group is $48 - 18 = 30$.

The ratio 5 : 3 is the same as the ratio 50 : 30. So in a group which has 30 girls, in which the ratio of boys to girls is 5 : 3, there will be 50 boys. The number of boys is 18. To make this up to 50, the number of boys that need to join the group is $50 - 18 = 32$.

18. In the addition sum shown, each letter represents a different non-zero digit.

What digit does X represent?

$$\begin{array}{cccc} & S & E & E \\ + & S & E & E \\ \hline A & X & E & S \end{array}$$

A 1 B 3 C 5 D 7 E 9

SOLUTION **D**

From the units column we see that either $E+E = S$, or there is a carry, in which case $E+E = 10+S$. However, from the tens column, we deduce that $E + E \neq S$. So there is a carry from the units column to the tens column.

Therefore, from the tens column we see that

$$1 + E + E = 10 + E.$$

If we subtract $E + 1$ from both sides of this equation, we deduce that $E = 9$. It now follows from the units column that $S = 8$. Hence the sum is

$$\begin{array}{cccc} & 8 & 9 & 9 \\ + & 8 & 9 & 9 \\ \hline 1 & 7 & 9 & 8 \end{array}$$

It follows that X represents 7.

FOR INVESTIGATION

18.1 Invent some problems about addition sums of a similar kind.

19. Three boxes under my stairs contain apples or pears or both. Each box contains the same number of pieces of fruit. The first box contains all twelve of the apples and one-ninth of the pears.

How many pieces of fruit are there in each box?

A 14 B 16 C 18 D 20 E 36

SOLUTION **B**

METHOD 1

The first box contains one-ninth of the pears. Therefore, the other two boxes contain, between them, eight-ninths of the pears. So, since they contain the same number of pieces of fruit, but no apples, they each contain four-ninths of the pears.

So the difference between the number of pears in each of the second and third boxes and the number of pears in the first box is equal to three-ninths of the number of pears, that is, to one-third of the number of pears.

Therefore, as the first box holds the same number of pieces of fruit as the other two boxes, the 12 apples in the first box match the extra one-third of the number of pears in the other two boxes. Since one-third of the pears amount to 12 pears, it follows that altogether there are 36 pears.

Hence the total number of pieces of fruit is 48. Since these are equally divided between the boxes, each of the three boxes contains 16 pieces of fruit.

METHOD 2

Since the first box contains one-ninth of the pears, we avoid fractions by supposing that there are $9p$ pears. It follows that there are p pears in the first box and altogether $8p$ pears in the other boxes. As they contain equal numbers of piece of fruit, there are $4p$ pears in each of these boxes.

Hence there are $12 + p$ pieces of fruit in the first box, and $4p$ in each of the other two boxes.

Since each box contains the same number of pieces of fruit

$$12 + p = 4p,$$

and hence

$$12 = 3p,$$

from which it follows that

$$p = 4.$$

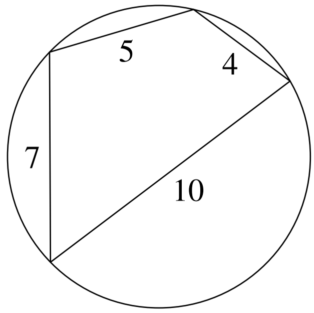
We have seen that there are $12 + p$ pieces of fruit in the first box. Since $p = 4$, we deduce that the number of pieces of fruit in each box is 16.

20. A cyclic quadrilateral has all four vertices on the circumference of a circle. Brahmagupta (598–670AD) gave the following formula for the area, A, of a cyclic quadrilateral whose edges have lengths a, b, c, d:

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$

where s is half of the perimeter of the quadrilateral.

What is the area of the cyclic quadrilateral with sides of length 4 cm, 5 cm, 7 cm and 10 cm?



A $6\,\text{cm}^2$ B $13\,\text{cm}^2$ C $26\,\text{cm}^2$ D $30\,\text{cm}^2$ E $36\,\text{cm}^2$

SOLUTION **E**

In Brahmagupta's formula, we have $a = 4$, $b = 5$, $c = 7$ and $d = 10$.

The total length of the perimeter, in cm, is $4 + 5 + 7 + 10 = 26$. Therefore $s = 26 \div 2 = 13$.

It follows that

$$\begin{aligned} s-a &= 13-4 = 9,\\ s-b &= 13-5 = 8,\\ s-c &= 13-7 = 6,\\ \text{and} \quad s-d &= 13-10 = 3. \end{aligned}$$

Hence

$$\begin{aligned} (s-a)(s-b)(s-c)(s-d) &= 9 \times 8 \times 6 \times 3\\ &= 9 \times 144. \end{aligned}$$

Therefore

$$\begin{aligned} A &= \sqrt{9 \times 144}\\ &= 3 \times 12\\ &= 36. \end{aligned}$$

Therefore the area of the cyclic quadrilateral is $36\,\text{cm}^2$.

FOR INVESTIGATION

20.1 Find a proof of Brahmagupta's formula for the area of a cyclic quadrilateral. [Either work out a proof for yourself, or look in a book or on the web, or ask your teacher.]

20.2 Heron's formula says that the area of a triangle with sides of lengths a, b and c is

$$\sqrt{s(s-a)(s-b)(s-c)},$$

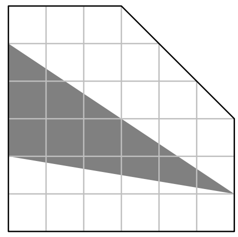
where s is half of the perimeter of the triangle. Use Brahmagupta's formula for the area of a cyclic quadrilateral to deduce Heron's formula for the area of a triangle.

20.3 Use Heron's formula to find the area of an equilateral triangle with sides of length 1.

21. The diagram shows a pentagon drawn on a square grid. All vertices of the pentagon and triangle are grid points.

What fraction of the area of the pentagon is shaded?

A $\frac{2}{7}$ B $\frac{1}{3}$ C $\frac{2}{5}$ D $\frac{1}{4}$ E $\frac{2}{9}$



SOLUTION **A**

We assume that the small squares making up the grid have a side length of 1 unit.

The pentagon consists of a 6×6 square, with half of a 3×3 square removed. Hence the area of the pentagon, in square units, is

$$6 \times 6 - \tfrac{1}{2}(3 \times 3) = 36 - \tfrac{9}{2} = \tfrac{63}{2}.$$

The area of a triangle is half the base multiplied by the height. The shaded triangle has a base of length 3 units and height 6 units. Therefore, its area, in square units, is

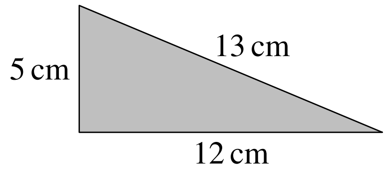
$$\tfrac{1}{2}(3 \times 6) = 9.$$

It follows that the fraction of the area of the pentagon that is shaded is

$$\frac{9}{\frac{63}{2}} = \frac{9 \times 2}{63} = \frac{2}{7}.$$

22. Four copies of the triangle shown are joined together, without gaps or overlaps, to make a parallelogram.

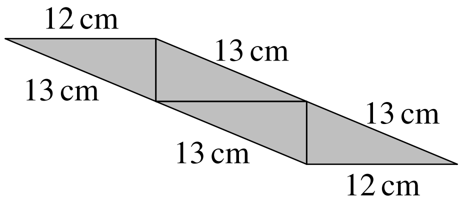
What is the largest possible perimeter of the parallelogram?



A 46 cm B 52 cm C 58 cm D 62 cm E 76 cm

Solution **E**

The four triangles between them have 12 edges. When they are fitted together there must be at least 3 places where two of the triangles meet along an edge. At each of these places 2 of the 12 edges form interior edges of the parallelogram. So the perimeter of the parallelogram will be made up of at most $12 - 3 \times 2 = 6$ of the edges of the triangles. So, if we can arrange the triangles so that the perimeter is made up of all four edges of length 13 cm, and two edges of length 12 cm, this would give us the largest possible perimeter.



The diagram shows that such an arrangement is possible. So the length of the longest possible perimeter is $4 \times 13\,\text{cm} + 2 \times 12\,\text{cm} = 76\,\text{cm}$.

For investigation

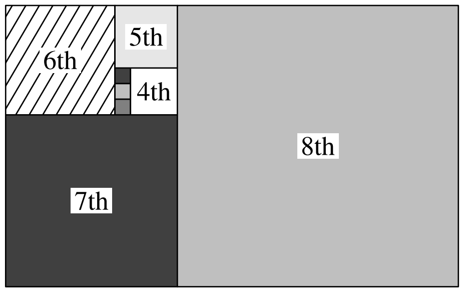
22.1 What is the smallest possible perimeter of a parallelogram that can be made by joining four copies of the given triangle without gaps or overlaps?

22.2 What is the largest possible perimeter of a parallelogram that can be made by joining six copies of the given triangle without gaps or overlaps?

22.3 What is the smallest possible perimeter of a parallelogram that can be made by joining six copies of the given triangle without gaps or overlaps?

22.4 Is it possible to make a parallelogram by joining an *odd* number of copies of the given triangle without gaps or overlaps?

23. The diagram shows the first few squares of a 'spiral' sequence of squares. All but the first three squares have been numbered.



After the first six squares, the sequence is continued by placing the next square alongside three existing squares – the largest existing square and two others.

The three smallest squares have sides of length 1.

What is the side length of the 12th square?

A 153 B 123 C 83 D 53 E 13

Solution **B**

We see from the diagram in the question that the side length of the 4th square is 3, and the side length of the 5th square is 4.

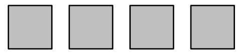
Also, the side length of the 6th square is the sum of the side lengths of the 4th and 5th squares. So the 6th square has side length $3 + 4 = 7$. Similarly the side length of the 7th square is the sum of the side lengths of the 5th and 6th squares. So the 7th square has side length $4 + 7 = 11$.

The sequence of side lengths continues in this way. After the first three numbers, each number in this sequence is the sum of the previous two numbers. Therefore the first 12 numbers in the sequence are

$$1, 1, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123.$$

We therefore see that the side length of the 12th square is 123.

24. Part of a wall is to be decorated with a row of four square tiles. Three different colours of tiles are available and there are at least two tiles of each colour available. Tiles of all three colours must be used.



In how many ways can the row of four tiles be chosen?

A 12 B 18 C 24 D 36 E 48

Solution **D**

We will suppose that the colours of the tiles that are available are red, blue and green.

We need to count the number of different ways in which we can choose to places the tiles, so that we use at least one tile of each of the three colours. To do this we will need two tiles of one colour and one tile of each of the other two colours.

The method we use is to split our choice of how to arrange the tiles into a sequence of independent choices, and to count the number of different ways in which we can make these choices.

The first choice is to decide the colour of the two tiles with the same colour. There are 3 ways to choose this colour. Once we have made this choices, we have no further choice of colours as we must have one tile of each of the two other colours.

We now need to count the number of different ways to place the four tiles in a row after we have chosen their colours.

Suppose, for the sake of argument, that we have decided to have two red tiles and hence one blue tile and one green tile.

We can decide how to place these tiles by first choosing the place for the blue tile, and then the place for the green tile. Then there will be no further choice, as we will need to put the two red tiles in the two remaining places.

There are 4 choices for the position of the blue tile.

Then, whichever position we choose for the blue tile, there will remain 3 choices of position for the green tile.

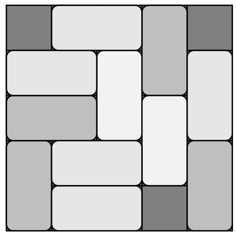
We thus see that placing the tiles involves three independent choices and that the number of possibilities for these choices are 3, 4 and 3 respectively.

Each of the 3 choices of colour shared by two of the tiles may be combined with any of the 4 choices of position for the tile of the second colour. So there are 3×4 ways to combine these choices.

Each of these 3×4 ways of making the first two choices may be combined with any of the 3 choices of position for the tile of the third colour.

Therefore the total number of different ways of combining these choices is $3 \times 4 \times 3 = 36$. Hence there are 36 different ways in which the row of four tiles can be chosen.

25. Beatrix places dominoes on a 5×5 board, either horizontally or vertically, so that each domino covers two small squares. She stops when she cannot place another domino, as in the example shown in the diagram.

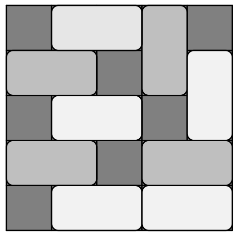


When Beatrix stops, what is the largest possible number of squares that may be still uncovered?

A 4 B 5 C 6 D 7 E 8

Solution **D**

The board has 25 squares. Each domino covers two squares. So, however many dominoes are placed on the board, the total number of squares that are covered will be even. So the number of uncovered squares will be odd. In particular, it is not possible to have 8 uncovered squares.



The diagram shows that it is possible for Beatrice to place 9 dominoes on the board in such a way that she cannot place another domino. This leaves 7 uncovered squares.

As this is the largest odd number given as one of the options, we conclude that D is the correct option.

Note

This is an adequate answer in the context of the JMC. However, for a complete answer we need to give a proof that we cannot place dominoes on the board so that there are more than 7 uncovered squares, and arranged in such a way so that it is not possible to add an additional domino. We do this using the method of *proof by contradiction*

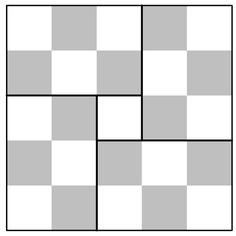
So we suppose that we have placed dominoes on the board so that there are more than 7 uncovered squares and it is not possible to place an additional domino on the board. Our aim is to show that this leads to a contradiction.

We have already noted that there must be an odd number of uncovered squares. So there are at least 9 uncovered squares.

For the purpose of this argument it is convenient to refer to an uncovered square as a *hole*. We also colour the squares of the board alternately black and white, as on a chessboard. We refer to holes on white squares as *white holes*, and those on black squares as *black holes*.

Each domino covers one white and one black square. Therefore, the dominoes cover equal numbers of white squares and black squares. The board has 13 white squares and 12 black squares. It follows that the number of white holes is one more than the number of black holes. So there are at least five white holes and at least four black holes.

Note that no two of these holes can have a side in common, as the presence of two adjacent holes would enable us to place an additional domino on the board. (Holes that have a side in common are said to be *adjacent*.)

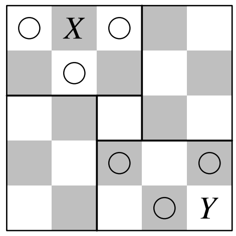


We first show that the centre square is a hole.

In the diagram on the right the 5×5 board has been divided into four 2×3 rectangles, and the centre square by itself.

There are at least 9 holes. Therefore, if the centre square is not a hole, there must be at least three holes in at least one of the 2×3 rectangles. In fact, we show that we cannot have as many as three holes in any of these rectangles.

In the diagrams below, we use ○ to indicate a square that is a hole.

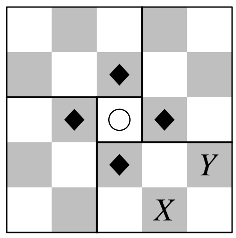


As no two holes can be adjacent, if there are three holes within a 2×3 rectangle, they must either all be white holes, as in the rectangle at top left of the diagram on the right, or all black holes, as in the rectangle in the bottom right.

In the first case, it would not be possible to place a domino on the square labelled X; in the second case, it would not be possible to place a domino on the square labelled Y. So, in either case, there would be adjacent holes. Hence this is impossible.

We therefore deduce that the central square must be a hole. It follows that the four black squares adjacent to the central square cannot be holes. However, as we have already seen, there must be at least four black holes somewhere on the board. Also, in addition to the hole in the centre square, there are at least four other white holes somewhere on the board.

We have already noted that the two black squares adjacent to a corner square cannot both be holes. It follows that in each of the four 2×3 rectangles exactly one of the black squares adjacent to each corner square is a hole. Since the white squares adjacent to this black hole cannot be a hole, there cannot be two white holes in any of the 2×3 rectangles. It follows that there is also exactly one white hole in each of these rectangles.



Consider the 2×3 rectangle in the bottom right corner of the diagram on the right. In this diagram ◆ is used to indicate a square that must be occupied by a domino.

The square labelled X cannot be a hole, as it would then not be possible to have a white hole in the rectangle.

Because the diagram is symmetric about the diagonal from top left to bottom right, the black square labelled Y also cannot be a hole. So there cannot be any black holes.

This is the contradiction we have been aiming at. We therefore conclude that we cannot place dominoes on the board so as to have more than 7 uncovered squares, and so that it is impossible to place an additional domino on the board.

This completes the proof.