UK INTERMEDIATE MATHEMATICAL CHALLENGE

THURSDAY 6TH FEBRUARY 2014

Organised by the **United Kingdom Mathematics Trust**
from the School of Mathematics, University of Leeds

http://www.ukmt.org.uk

SOLUTIONS LEAFLET

This solutions leaflet for the IMC is sent in the hope that it might provide all concerned with some alternative solutions to the ones they have obtained. It is not intended to be definitive. The organisers would be very pleased to receive alternatives created by candidates. More comprehensive solutions are on the website at: http://www.ukmt.org.uk/

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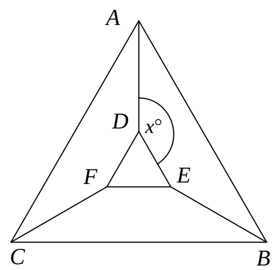
1. **A** 25% of $\frac{3}{4} = \frac{1}{4} \times \frac{3}{4} = \frac{3}{16}$.

2. **D** The first four options are the smallest positive integers which are both odd and not prime. However, the next largest odd numbers after 9, 15, 21 are 11, 17, 23 respectively and these are all prime. The next largest odd number after 25 is 27, which is not prime. So 25 is the smallest positive integer which satisfies all three conditions.

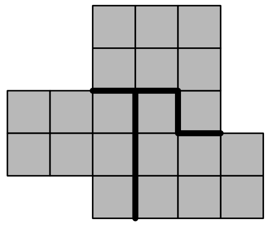
3. **E** Clearly AD lies along one of the lines of symmetry of the figure. So $\angle FDA = \angle EDA = x°$. Triangle DEF is equilateral so $\angle EDF = 60°$.
The angles which meet at a point sum to 360°, so $x + x + 60 = 360$.
Therefore $x = 150$.



4. **C** Since m is even, $m = 2k$ for some integer k. So $3m + 4n = 2(3k + 2n)$; $5mn = 2(5kn)$; $m^3n^3 = 8k^3n^3$ and $5m + 6n = 2(5k + 3n)$, which are all even. As n is odd, $3n$ is also odd. So $m + 3n$ is an even integer plus an odd integer and is therefore odd. The square of an odd integer is odd so $(m + 3n)^2$ is odd.

5. **E** In one complete cycle of 4 hours, the clock is struck $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$ times. So in 24 hours the clock is struck $6 \times 36 = 216$ times.

6. **E** The large shape consists of 21 small squares, so the required shape is made up of 7 small squares. So A and C may be eliminated. The diagram on the right shows that shape E is as required. It is left to the reader to check that neither B nor D was the shape used.

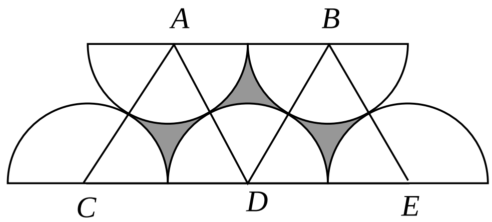


7. **B** Since 6 and 15 are factors of the integer, its prime factors will include 2, 3 and 5. So 10 and 30 will also be factors of the required integer. Seven of its factors are now known and as 1 must also be a factor, the required integer is 30, the factors of which are 1, 2, 3, 5, 6, 10, 15, 30.
(*Positive integers with exactly 8 factors are of the form pqr or pq^3 or p^7 where p, q, r are distinct primes*.)

8. **C** The missing die, if correctly placed in the figure, would show faces 1, 3, 5 placed in a clockwise direction around the nearest corner. An examination of each of the five proposed dice shows that only C has this property.

9. **A** Gill's car uses $p/100$ litres of petrol for every one kilometre travelled. So for a journey of length d km, $pd/100$ litres of petrol are required.

10. **B** A, B, C, D, E are the centres of the five semicircles. Note that AC joins the centres of two touching semicircles and therefore passes through the point of contact of the semicircles. So AC has length $2 + 2 = 4$. This also applies to all of the other sides of triangles ACD and BED. Hence both triangles are equilateral. So each of the nine arcs which make up the perimeter of the shaded shape subtends an angle of 60° at the centre of a semicircle.
So the length of the perimeter of the shaded figure is $9 \times \frac{1}{6} \times 2 \times \pi \times 2 = 6\pi$.



11. **C** Precisely one of Jenny and Willie is telling the truth since the number of people is either even or odd. Similarly, precisely one of Sam and Mrs Scrubitt is telling the truth since the number of people is either a prime number or a number which is the product of two integers greater than one. So although it is not possible to deduce who is telling the truth, it is possible to deduce that exactly two of them are doing so.

12. **D** Let the width of each strip be 1 unit. Then the triangle has base 8 and perpendicular height 8. So its area is equal to $\frac{1}{2} \times 8 \times 8 = 32$. Looking from the right, the area of the first shaded strip is 1 unit of area less than the first unshaded strip. This difference of 1 unit also applies to the other three pairs of strips in the triangle, which means that the shaded area is 4 less than the unshaded area. So the total shaded area is $\frac{1}{2}(32 - 4) = 14$. Therefore the required fraction is $\frac{14}{32} = \frac{7}{16}$.

13. B The smallest such number is $1 + 2 = 3$, whilst the largest is $99 + 100 = 199$. Every number between 3 and 199 may be written as $1 + n$ with $n = 2, 3, \ldots, 99$ or as $100 + n$ with $n = 1, \ldots, 99$. So in total there are $(199 - 3) + 1 = 197$ such numbers.

14. B Chris Froome's average speed $\approx \frac{3400}{84}$ km/h $\approx \frac{3400}{85}$ km/h $= \frac{200}{5}$ km/h = 40 km/h.

15. E Let Zac's number be x. Then $\frac{1}{2}x + 8 = 2x - 8$. So $x + 16 = 4x - 16$. Therefore $32 = 3x$, that is $x = 10\frac{2}{3}$.

16. C If the areas of the original and new triangles are the same then the product of the base and the perpendicular height must be the same for the two triangles. When the base of the original triangle is increased by 25%, its value is multiplied by $\frac{5}{4}$. So if the area is to remain unchanged then the perpendicular height must be multiplied by $\frac{4}{5}$, which means that its new value is 80% of its previous value. So it is decreased by 20%.

17. D The number of minutes in one week is $7 \times 24 \times 60$, which may be written as $7 \times (6 \times 4) \times (5 \times 3 \times 2 \times 2) = (7 \times 6 \times 5 \times 4 \times 3 \times 2) \times 2$. So the number of weeks in $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ minutes is $8 \div 2 = 4$.

18. B The point (m, n) is hidden if and only if m and n share a common factor greater than 1. So (6, 2) is hidden by (3, 1) since 6 and 2 have common factor 2. Also (6, 3) is hidden by (2, 1) whilst (6, 4) is hidden by (3,2). However, 6 and 5 have no common factor other than 1 and therefore (6, 5) is not a hidden point.

19. C Note that $8^m = (2^3)^m = 2^{3m} = (2^m)^3$ and $27 = 3^3$; so $2^m = 3$. Therefore $4^m = 2^m \times 2^m = 9$.

20. D Each exterior angle of a regular pentagon is $\frac{1}{5} \times 360° = 72°$. So each of the five circular arcs has radius 2 and so subtends an angle of $(180 + 72)°$ at a vertex of the pentagon. Therefore the area of each of the five shaded major sectors is $\frac{252}{360} \times \pi \times 2^2 = \frac{7}{10} \times \pi \times 4 = \frac{14\pi}{5}$. So the total shaded area is 14π.

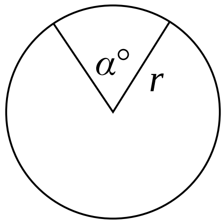
21. D Firstly suppose that any two knights X and Y win x and y bouts respectively and that x is at least as large as y. The difference between their total scores would be the same as if X had won $x - y$ bouts and Y had won none, since each of the separate totals would have been reduced by the same amount, namely $20y$. A similar procedure applies to losses. For example, if X won 3 and lost 6, while Y won 8 and lost 2, the difference between their total scores is the same as if X won 0 and lost 4, while Y won 5 and lost 0. In each case the difference is 32.

This argument shows that, in the case of the Black Knight, B, and the Red Knight, R, the smallest number of bouts will be achieved when one of B, R wins all his bouts and the other loses all his bouts. Also B has to score one more point than R. The possible scores for the knight who wins all his bouts are 20, 40, 60, 80, 100, 120, … . while the possible scores for the knight who loses all his bouts are 17, 34, 51, 68, 85, 102, 119, 136, … .

The first two numbers to differ by 1 are 119 and 120. Thus the Black Knight has a total of 120 corresponding to winning all of his 6 bouts and the Red Knight has a total of 119 corresponding to losing all of his 7 bouts.

22. B Let $a = a_1 a_2$ where a_2 is the largest square dividing a. Note that a_1 is then a product of distinct primes. Similarly write $b = b_1 b_2$ and $c = c_1 c_2$. Since ab is a square, $a_1 b_1$ must be a square; so $a_1 = b_1 = k$ say. Similarly $c_1 = k$. The smallest possible value of k is 2 (since a is not a square); and the smallest possible values for a_2, b_2, c_2 are 1, 4 and 9 in some order. This makes $a + b + c = 2 + 8 + 18 = 28$.

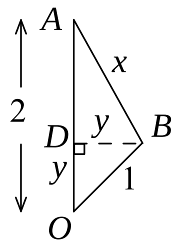
23. A Let the radius of the circle be r and let the angle of the sector be $\alpha°$.



Then the perimeter of the sector is $2r + \frac{\alpha}{360} \times 2\pi r$.
This equals $2\pi r$, the circumference of the original circle.
So $2r + \frac{\alpha}{360} \times 2\pi r = 2\pi r$.
Therefore the fraction of the area of the disc removed is
$\frac{\alpha}{360} = \frac{2\pi r - 2r}{2\pi r} = \frac{\pi - 1}{\pi}$.

24. A There are 9000 four-digit integers. To calculate the number of these which have four different digits, we note that we have a choice of 9 for the thousands digit. We now have a choice of 9 for the hundreds digit (since we can choose 0 as a possible digit). After these two digits have been chosen, we have a choice of 8 for the tens digit and then 7 for the units digit. So the number of four-digit numbers in which all digits are different is $9 \times 9 \times 8 \times 7$.
Therefore the number of four-digit numbers which have at least one digit repeated is $9000 - 9 \times 9 \times 8 \times 7 = 9(1000 - 9 \times 8 \times 7) = 9 \times 8 \times (125 - 9 \times 7) = 72 \times (125 - 63) = 72 \times 62$.

25. E Let each side of the octagon have length x. The octagon may be divided into eight triangles by joining the centre of the circle to the vertices of the octagon. One such triangle is shown. Each of these triangles has one side of length 1 (the radius of the smaller circle), one side of length 2 (the radius of the larger circle) and one side of length x. So all eight triangles are congruent. Therefore $\angle AOB = 360° \div 8 = 45°$.



Let D be the foot of the perpendicular from B to AO. Then triangle BDO is an isosceles right-angled triangle.
Let $OD = DB = y$. Applying Pythagoras' Theorem to triangle BDO:
$y^2 + y^2 = 1$. So $y = \frac{1}{\sqrt{2}}$.
Applying Pythagoras' Theorem to triangle ADB:
$x^2 = (2 - y)^2 + y^2 = \left(2 - \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 4 - 2 \times 2 \times \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2} = 5 - \frac{4}{\sqrt{2}} = 5 - 2\sqrt{2}$.
So the length of the perimeter is $8x = 8\sqrt{5 - 2\sqrt{2}}$.
(*Note that the length of AB may also be found by applying the Cosine Rule to triangle OAB.*)