UK INTERMEDIATE MATHEMATICAL CHALLENGE

THURSDAY 3rd FEBRUARY 2011

Organised by the **United Kingdom Mathematics Trust from the School of Mathematics, University of Leeds**

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The Actuarial Profession
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SOLUTIONS LEAFLET

This solutions leaflet for the IMC is sent in the hope that it might provide all concerned with some alternative solutions to the ones they have obtained. It is not intended to be definitive. The organisers would be very pleased to receive alternatives created by candidates.

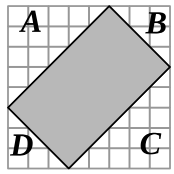
1. **B** $4.5 \times 5.5 + 4.5 \times 4.5 = 4.5(5.5 + 4.5) = 4.5 \times 10 = 45$.

2. **A** The diameter is $(5 \times 0.3 + 2.1)$ mm $= (1.5 + 2.1)$ mm $= 3.6$ mm.

3. **E** $12 \div 3 = 4$, but this is the example given in the question. $23 \div 4 \neq 5$; $34 \div 5 \neq 6$; $45 \div 6 \neq 7$. However, $56 \div 7 = 8$, so $s = 8$.
{*Note also that* $67 \div 8 \neq 9$.}

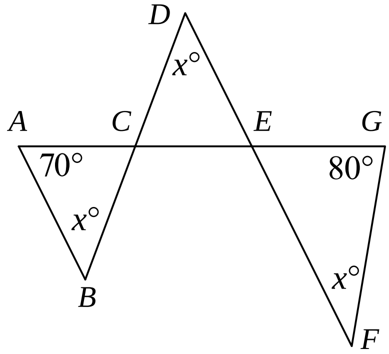
4. **E** The difference between the angles is $\left(\frac{5}{10} - \frac{2}{10}\right) \times 180°$.

5. **C** Triangles A and C are each of area $\frac{1}{2} \times 5 \times 5$ cm^2.
Triangles B and D are each of area $\frac{1}{2} \times 3 \times 3$ cm^2.
So the shaded area is $[64 - (25 + 9)]$ cm^2 $= 30$ cm^2.



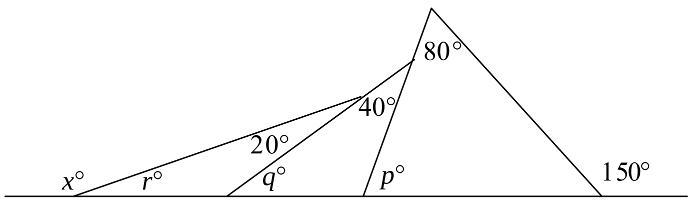
6. **B** The next palindromic number after 24942 is 25052, so the car travelled 110 miles in the two days.

7. **A** Alternate angles BDF and DFG are equal, so lines BD and FG are parallel. Therefore $\angle BCA = \angle FGC = 80°$ (corresponding angles). Consider triangle ABC: $x + 70 + 80 = 180$, so $x = 30$.



8. **E** The base of the open box is a square. Let its side be of length x cm. Then the total surface area of the box in cm^2 is $x^2 + 4 \times 2x = x^2 + 8x$. Hence $x^2 + 8x = 180$, that is $x^2 + 8x - 180 = 0$. Therefore $(x + 18)(x - 10) = 0$, which gives $x = -18$ or $x = 10$. As x is positive, it may be deduced that the open box has dimensions 10 cm × 10 cm × 2 cm. So its volume is 200 cm^3.

9. **A** In a triangle, an exterior angle is equal to the sum of the two interior, opposite angles. Repeatedly applying this theorem:
$p = 150 - 80 = 70$;
$q = p - 40 = 30$; $r = q - 20 = 10$.
Therefore $x = 180 - r = 170$.



10. **E** One tonne = 1000kg = 1 000 000g. So the number of mice is
6 000 000 ÷ 20 = 300 000.

11. **E** $19\frac{1}{2} \times 20\frac{1}{2} = \left(20 - \frac{1}{2}\right) \times \left(20 + \frac{1}{2}\right) = 20^2 - \left(\frac{1}{2}\right)^2 = 400 - \frac{1}{4} = 399\frac{3}{4}$.

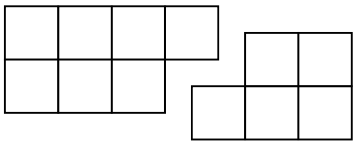
12. **D** $20 \div 11 = 1\frac{9}{11} = 1.818181\ldots$. So the first 2011 digits are 1006 '1's and 1005 '8's. Therefore the required total is $1006 \times 1 + 1005 \times 8 = 1006 + 8040 = 9046$.

13. **D** After the first mouse has eaten, $\frac{2}{3}$ of the cheese remains. After the second mouse has eaten, $\frac{2}{3}$ of $\frac{2}{3}$, that is $\frac{4}{9}$, of the cheese remains. Finally, after the third mouse has eaten, $\frac{2}{3}$ of $\frac{4}{9}$, that is $\frac{8}{27}$, of the cheese remains. So the mice ate $\frac{19}{27}$ of the cheese.

14. E $3 = \frac{2 + 2^2}{2}$; $10 = \frac{4 + 4^2}{2}$; $15 = \frac{5 + 5^2}{2}$; $21 = \frac{6 + 6^2}{2}$. However $\frac{7 + 7^2}{2} = 28$ and $\frac{8 + 8^2}{2} = 36$, so 30 is not exactly halfway between a positive integer and its square. (*Note that every number which is exactly halfway between a positive integer and its square is a triangle number. Can you explain why this is so*?)

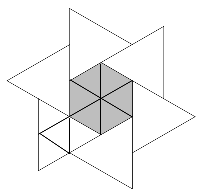
15. A In each rotation which C makes, the radius of the arc it describes is 1 unit. In the first rotation, C turns through an angle of 120°, so it moves a distance $\frac{1}{3} \times 2 \times \pi \times 1$, that is $\frac{2\pi}{3}$. As it is the centre of the second rotation, C does not move during it. In the third rotation, C again turns through an angle of 120°, so the total distance travelled is $2 \times \frac{2\pi}{3} = \frac{4\pi}{3}$.

16. C The L-shape needs to be divided as shown since neither of the pieces is to be a square. Notice that one of the pieces must be turned over. The difference between the areas of the two pieces is $7 - 5 = 2$.

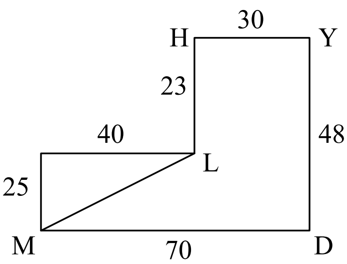


17. C The reduction of 15% off sale prices is equal to a reduction of 7.5% off the original prices. Therefore the total reduction on the original prices is $(50 + 7.5)\% = 57.5\%$.

18. D As the diagram shows, each equilateral triangle may be divided into four equilateral triangles of side 1, whilst the hexagon may be divided into six equilateral triangles of side 1. Therefore the fraction of the whole shape which is shaded is $\frac{6}{6 \times 4 + 6} = \frac{6}{30} = \frac{1}{5}$.



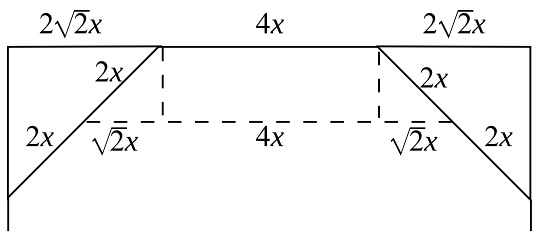
19. B As can be seen from the diagram, Manchester is 40km west of Leeds and 25 km south of it. Therefore, by Pythagoras' Theorem, the distance in km from Leeds to Manchester as the crow flies is $\sqrt{25^2 + 40^2} = 5\sqrt{5^2 + 8^2} = 5\sqrt{89}$. Now $\sqrt{81} = 9$ and $\sqrt{100} = 10$, so $9 < \sqrt{89} < 10$. This means that the required distance is between 45km and 50km and, of the options given, only 47km (corresponding to the approximation $\sqrt{89} \approx 9.4$) lies in this interval. (*Please note that the distances given in this problem are all approximate*.)



All distances are in km

20. D Let Max and Molly meet after the latter has travelled x miles. Then Max has travelled $(6 - x)$ miles. So $6 - x = 2x$, thus $x = 2$. Therefore Molly walks a total of 4 miles.

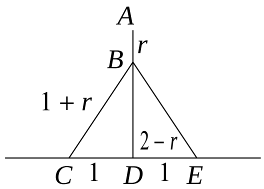
21. B Let $4x$ be the length of each side of the regular octagon. The diagram shows part of the figure. The four triangles shown in the diagram are all isosceles right-angled triangles. In such triangles the ratio of the length of the hypotenuse to the length of the shorter sides is $\sqrt{2} : 1$.



So in the larger triangles which have hypotenuse of length $4x$, the length of the shorter sides is $2\sqrt{2}x$, whilst the smaller triangles with hypotenuse $2x$ have shorter sides of length $\sqrt{2}x$.
Therefore the shaded square in the question has side of length $(4 + 2\sqrt{2})x$.
The length of the side of the outer square is $(4\sqrt{2} + 4)x = \sqrt{2}(4 + 2\sqrt{2})x$.
Therefore the two squares have sides in the ratio $1 : \sqrt{2}$, which means that their areas have ratio $1 : 2$.

22. B $5^p = 9$ and $9^q = 12$. Therefore $(5^p)^q = 12$, that is $5^{pq} = 12$. Similarly, as $12^r = 16$ then $5^{pqr} = 16$, as $16^s = 20$ then $5^{pqrs} = 20$ and, finally, as $20^t = 25$ then $5^{pqrst} = 25$. Therefore $pqrst = 2$.

23. A Points A, B, C, D and E are, respectively: the point where the large semicircle and the circle touch, the centre of the circle, the centre of the left-hand semicircle, the centre of the large semicircle and the centre of the right-hand semicircle. The radius of the circle is r m. In triangle BCD, CD has length 1m, BD has length $(2 - r)$ m, since AD is a radius of the semicircle of diameter 4m, and BC has length $(1 + r)$ m, since it is the sum of the radii of the left-hand semicircle and the circle. Therefore, by Pythagoras' Theorem: $(1 + r)^2 = 1^2 + (2 - r)^2$, that is $1 + 2r + r^2 = 1 + 4 - 4r + r^2$. So $6r = 4$ and the radius of the circle is $\frac{2}{3}$m.



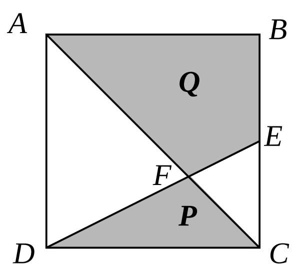
(*Note that triangle BCD is a* 3,4,5 *triangle with sides* $\frac{3}{3}$m, $\frac{4}{3}$m *and* $\frac{5}{3}$m.)

24. C Firstly we give examples to show that Paul's answer could have been any of A, B, D or E.

A: If n is prime then the only factor of n other than itself is 1.
B: Take $n = 4$. Its factors are 1, 2 and 4, and $1 + 2 = 3$.
D: Take $n = 8$. Its factors are 1, 2, 4 and 8, and $1 + 2 + 4 = 7$.
E : Take $n = 15$. Its factors are 1, 3, 5 and 15, and $1 + 3 + 5 = 9$.

We now show that Paul's answer cannot be C. If the sum were 5, then the factors of n other than itself would have to be 1 and 4, as we are not allowed to repeat any number in the sum. However, if 4 is a factor of n, then 2 is also a factor, which produces a contradiction.

25. D Let triangle CEF have area a. Note that $\angle AFD = \angle CFE$ (vertically opposite angles) and $\angle DAF = \angle ECF$ (alternate angles), so triangles ADF and CEF are similar.
Note also that the side AD is twice the length of the corresponding side CE.



Hence:
(i) triangle ADF has area $4a$ and;
(ii) AF has twice the length of the corresponding side CF.
View AF and CF as bases of the triangles ADF and CDF (which then share the same height). Therefore, by (ii), triangle ADF has twice the area of triangle CDF (area $\boldsymbol{P}$), which thus is $2a$.
The area of triangle ACD is $6a$; so that of triangle ABC is also $6a$ and that of area $\boldsymbol{Q}$ is $5a$. So the required ratio is $2 : 5$.