United Kingdom
Mathematics Trust

IMOK Olympiad

Organised by the United Kingdom Mathematics Trust

Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

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Cayley

1. Each of Alice and Beatrice has their birthday on the same day.

In 8 years' time, Alice will be twice as old as Beatrice. Ten years ago, the sum of their ages was 21.

How old is Alice now?

Solution

Let Alice be a years old now and let Beatrice be b years old now.

In 8 years' time, Alice will be twice as old as Beatrice, so that

$$a + 8 = 2(b + 8),$$

which we may simplify to

$$a - 2b = 8. \qquad (1)$$

Ten years ago, the sum of Alice's age and Beatrice's age was 21, so that

$$a - 10 + b - 10 = 21,$$

which we may rewrite in the form

$$a + b = 41. \qquad (2)$$

Adding equation (1) to 2 × equation (2) in order to eliminate b, we get $3a = 90$, so that $a = 30$.

Hence Alice is now 30 years old.

2. In the addition shown, each of the letters D, O, G, C, A and T represents a different digit.

What is the value of $D + O + G + C + A + T$?

$$\begin{array}{r} D\,O\,G \\ +\,C\,A\,T \\ \hline 1\,0\,0\,0 \end{array}$$

Solution

We first note that when we add two different single digits, the result is between $0 + 1$ and $8 + 9$, that is, the sum is between 1 and 17.

Starting with the 'ones' column of the addition in the question, G and T add to give a result that is between 1 and 17 and with 0 as its 'ones' digit, so their total is 10, that is,

$$G + T = 10. \tag{1}$$

Moving to the 'tens' column, we have to include the 1 carried from the 'ones' column so that

$$O + A + 1 = 10,$$

that is,

$$O + A = 9 \tag{2}$$

and therefore 1 is carried forward.

Similarly, in the 'hundreds' column, we include the 1 carried forward and have

$$D + C + 1 = 10$$

so that

$$D + C = 9. \tag{3}$$

Summing the three equations (3) to (1), we obtain $D + O + G + C + A + T = 28$.

Is a solution possible?

In the diagram alongside, for example, $D + O + G + C + A + T = 28$, as required.

$$\begin{array}{r} 321 \\ +679 \\ \hline 1000 \end{array}$$

3. The triangle ABC is isosceles with $AB = BC$. The point D is a point on BC, between B and C, so that $AC = AD = BD$.

What is the size of angle ABC?

Solution

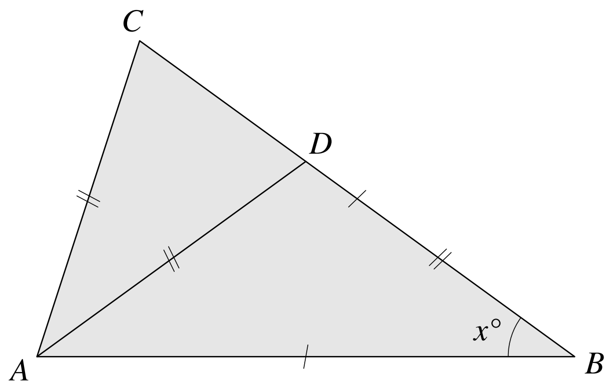
Commentary

For this solution, we use the results that

(a) 'the angles opposite the two equal sides of an isosceles triangle are equal' (base angles of an isosceles triangle),

(b) 'an exterior angle of a triangle is equal to the sum of the two opposite interior angles' (exterior angle of a triangle),

(c) 'the angle sum of a triangle is 180° ' (angle sum of a triangle).



Let angle ABC be $x°$, as shown.

Triangle ABD is isosceles since $AD = BD$, so that $\angle ABD = \angle DAB = x°$ (base angles of an isosceles triangle).

Thus, from triangle ABD, we have $\angle ADC = 2x°$ (exterior angle of a triangle).

Triangle ACD is isosceles with $AC = AD$ so that $\angle ADC = \angle ACD = 2x°$ (base angles of an isosceles triangle).

Also, triangle ABC is isosceles with $AB = BC$ so that $\angle BCA = \angle CAB = 2x°$ (base angles of an isosceles triangle).

Now, considering triangle ABC, we have $2x + x + 2x = 180$ (angle sum of a triangle).

Hence $x = 36$ and so angle ABC is 36°.

4. Arrange the digits 1, 2, 3, 4, 5, 6, 7, 8 to form two 4-digit integers whose difference is as small as possible.

Explain clearly why your arrangement achieves the smallest possible difference.

SOLUTION

Let the two numbers be '$abcd$' and '$efgh$', with the smaller being '$abcd$'.

For a difference of less than 1000,

$$e = a + 1, \qquad (*)$$

so that the two 4-digit numbers are in consecutive thousands. All other choices have a difference of more than 1000.

For '$efgh$' to be as little above '$e000$' as possible, we use 1, 2 and 3 for f, g and h respectively.

Similarly, for'$abcd$' to be as close as possible to, but below, '$e000$' we use 8, 7 and 6 for b, c and d respectively.

This leaves 4 and 5, which satisfy (*), to be the values of a and e respectively.

Therefore the smallest difference comes from $5123 - 4876$, which is 247.

5. Howard chooses n different numbers from the list 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, so that no two of his choices add up to a square.

What is the largest possible value of n?

Solution

> Commentary
>
> When compiling the list of pairs which add up to a square, we can either work through the list (what numbers pair with 2?, what pairs with 3?, ...), or note that we are looking for squares between $2 + 3$ and $10 + 11$, that is, 9 and 16.

When considering which numbers Howard should choose, we have to avoid the following pairs of numbers since their sum is a square: 2 and 7, 3 and 6, 4 and 5, 5 and 11, 6 and 10, and 7 and 9.

By considering, for example, the first three pairings, no two of which have a number in common, we see that there is no group of numbers with less than three members with the property that one of the numbers in the group is contained in each of the above six pairings.

However, each of the above six pairings contains one of the three numbers 5, 6 or 7.

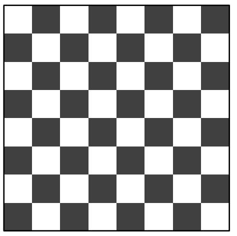
> Commentary
>
> Not choosing 5, 6 and 7 means that one of the numbers from all six pairings has been left out, so each pairing will not be included.
>
> So, for the largest possible list, Howard should choose 2, 3, 4, 8, 9, 10 and 11.

Thus the smallest number of individual numbers that need to be avoided so that none of these pairings is present is three.

Hence the largest possible value of n is seven.

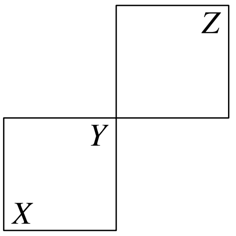
6. A chessboard is formed from an 8×8 grid of alternating black and white squares, as shown. The side of each small square is 1 cm.

What is the largest possible radius of a circle that can be drawn on the board in such a way that the circumference is entirely on white squares or corners?



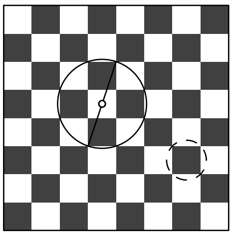
Solution

To go from one white square to another, it is necessary to pass through a corner.



But it is not possible for a circle to pass through the point where two white squares meet (such as Y in the diagram alongside) and both the diagonally opposite corners of this point (X and Z), since these points form a straight line and not an arc of a circle.

Is it possible for a circle to pass through exactly one of X and Z, for *every* choice of Y on the circle? In other words, the circle is alternately "close" to a diagonal and an edge of white squares.



The diagram alongside shows one such circle, whose centre is as shown. The circle has diameter $\sqrt{1^2 + 3^2}$, from Pythagoras' theorem.

A circle passing through Y and neither X nor Z, for *every* point Y on the circle, such as the dashed circle in the diagram alongside, has a smaller radius.

Therefore the largest possible radius of the circle is $\frac{1}{2}\sqrt{10}$.

Hamilton

1. A number of couples met and each person shook hands with everyone else present, but not with themselves or their partners.

There were 31 000 handshakes altogether.

How many couples were there?

Solution

Suppose that there were c couples. Then each couple interacted with $c - 1$ other couples (every couple except themselves).

Then $c \times (c - 1)$ gives double the total number of interactions between couples (since we have counted each interaction exactly twice – once for each couple involved in it). So the total number of interactions is $\frac{1}{2}c(c - 1)$.

Each interaction between couples results in four handshakes, so the total number of handshakes was $4 \times \frac{1}{2}c(c - 1)$, that is, $2c(c - 1)$.

Therefore $2c(c - 1) = 31\,000$, so that $c^2 - c - 15\,500 = 0$. Hence $(c - 125)(c + 124) = 0$, so that $c = 125$ (ignoring the negative answer, which is not practicable here).

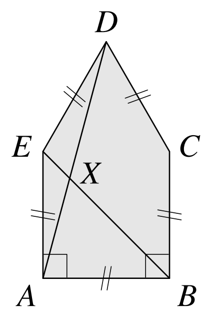
Thus there were 125 couples.

Note

If you have not yet seen how to solve a quadratic equation, you could also notice that you are looking for two consecutive numbers which multiply to 15 500. Since one of these needs to be a multiple of 25 (can you see why?), completing a methodical search is not too onerous and, if explained thoroughly, is just as valid a solution.

2. The diagram shows a pentagon $ABCDE$ in which all sides are equal in length and two adjacent interior angles are $90°$. The point X is the point of intersection of AD and BE.

Prove that $DX = BX$.



Solution

AE and BC are equal, and are parallel because the angles at A and B are both equal to $90°$ (allied angles, converse). Hence $ABCE$ is a parallelogram (opposite sides equal and parallel), so that EC and AB are equal (property of a parallelogram).

It follows that $ABCE$ is a square (a rhombus with a right angle), and that triangle CDE is equilateral (all sides equal).

Now triangle ADE is isosceles, so that $\angle ADE = \angle DAE$ (base angles of an isosceles triangle). However, $\angle DEA = 60° + 90° = 150°$, so that $\angle ADE = \angle DAE = 15°$ (angle sum of a triangle). Similarly, triangle BCD is isosceles, and$\angle CDB = \angle DBC = 15°$.

Therefore $\angle BDX = 60° - 15° - 15° = 30°$.

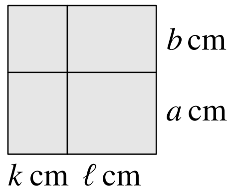
Also $\angle XBC = 45°$ (a diagonal of a square bisects the angle), so that $\angle XBD = 45° - 15° = 30°$.

Hence $\angle XBD = \angle BDX$, so $DX = BX$ (sides opposite equal angles).

3. A 4 cm × 4 cm square is split into four rectangular regions using two line segments parallel to the sides.

How many ways are there to do this so that each region has an area equal to an integer number of square centimetres?

Solution



Let the lengths of the sides of the regions be k cm, ℓ cm, a cm and b cm, as shown. We know that each of these is less than 4 cm.

Then the regions have areas ka cm^2, kb cm^2, ℓa cm^2 and ℓb cm^2 respectively, and thus each of ka, kb, ℓa and ℓb is an integer.

Hence $ka + kb = k(a + b) = 4k$ is an integer, so that k is an integer multiple of $\frac{1}{4}$. Similarly, each of ℓ, a and b is an integer multiple of $\frac{1}{4}$.

However, none of k, ℓ, a and b is an integer multiple of 4, because each is between 0 and 4. Hence each of them has to be an integer multiple of $\frac{1}{2}$ in order that each of ka, kb, ℓa and ℓb is an integer.

If both a and k are integers

then clearly all four rectangles have integer area. In this case we have three choices for a (1, 2 and 3) and three choices for k (also 1, 2 and 3), giving $3 \times 3 = 9$ choices.

If one of a or k is not an integer (that is, an odd multiple of $\frac{1}{2}$)

then the other is 2, otherwise ka is not an integer. When $a = 2$ it is possible for k to take four such values $\left(\frac{1}{2}, \frac{3}{2}, \frac{5}{2} \text{ and } \frac{7}{2}\right)$ and similarly when $k = 2$, a could take the same four values.

Note that once a and k have been chosen, then these determine b and ℓ uniquely.

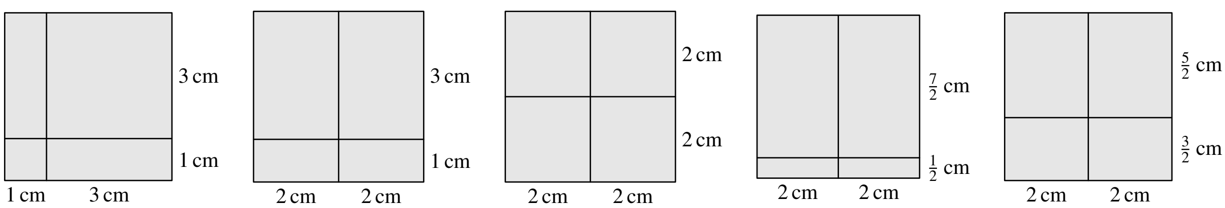
Thus altogether there are $9 + 4 + 4 = 17$ ways.

Note

If the grid can be rotated or reflected, so that, for example, $a = 1$ is counted as being the same as $k = 1$ or $a = 3$, then there are 5 ways, given by the following pairs of values of (a, k) or (k, a):

$$(1, 1), (1, 2), (2, 2), \left(\tfrac{1}{2}, 2\right), \left(\tfrac{3}{2}, 2\right),$$

as shown below.



4. Each of A and B is a four-digit palindromic integer, C is a three-digit palindromic integer, and $A - B = C$.

What are the possible values of C?

[*A palindromic integer reads the same 'forwards' and 'backwards'.*]

Solution

Let the integers be A = '$adda$', B = '$beeb$' and C = 'cfc'. Then we may rewrite $A - B = C$ in the form of the addition sum shown.

$$\begin{array}{r} b\,e\,e\,b \\ +\,c\,f\,c \\ \hline a\,d\,d\,a \end{array}$$

Note that none of a, b and c can be zero, because A, B and C are integers with 4, 4 and 3 digits.

Considering the 'ones' column, $a \neq b$ since $c \neq 0$. Considering the 'thousands' column, since $a \neq b$, there is a 'carry' from the 'hundreds' column. It follows that $a = b + 1$, so that, from the 'ones' column, $c = 1$. Once we know that $c = 1$, the only way there can be a 'carry' from the 'hundreds' to the 'thousands' column is to have $e = 9$ or $e = 8$.

There are two cases.

$e = 9$

In this case, we have the addition shown alongside.

$$\begin{array}{r} b\,9\,9\,b \\ +\,1\,f\,1 \\ \hline a\,d\,d\,a \end{array}$$

Considering the 'hundreds' column, d is either 0 or 1, depending on whether there is not, or is, a 'carry' from the 'tens' column. But when $d = 0$ there *is* such a 'carry' ($f = 1$), which leads to a contradiction.

Hence $d = 1$ and thus $f = 2$, as shown alongside.

$$\begin{array}{r} b\,9\,9\,b \\ +\,1\,2\,1 \\ \hline a\,1\,1\,a \end{array}$$

$e = 8$

We now have the addition shown alongside.

$$\begin{array}{r} b\,8\,8\,b \\ +\,1\,f\,1 \\ \hline a\,d\,d\,a \end{array}$$

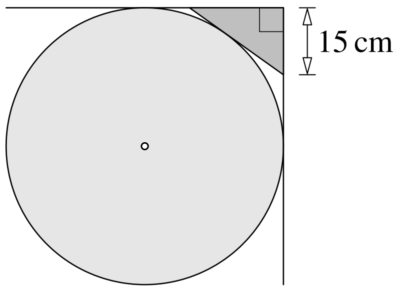
In this case, we similarly have $d = 0$ and $f = 2$, as shown alongside.

$$\begin{array}{r} b\,8\,8\,b \\ +\,1\,2\,1 \\ \hline a\,0\,0\,a \end{array}$$

Therefore, in either case, there is only one possible value of C, namely 121. An important step is to check whether it is actually possible to find values of A and B which give $C = 121$, and working on from one of the above cases, it is easy to find an example, such as $2112 - 1991 = 121$.

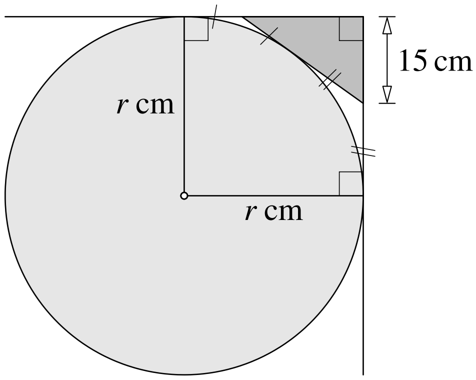
5. The area of the right-angled triangle in the diagram alongside is 60 cm^2. The triangle touches the circle, and one side of the triangle has length 15 cm, as shown.

What is the radius of the circle?



Solution

The area of a triangle is $\frac{1}{2}$base × height, so the triangle has sides of length 8 cm, 15 cm and 17 cm (using Pythagoras' theorem).



Let the radius of the circle be r cm.

Draw two radii, as shown. Since a tangent meets the radius at the point of contact at right angles, we have a square, with sides of length r cm.

Then, using the equal tangent theorem, we have

$$(r-8)+(r-15)=17,$$

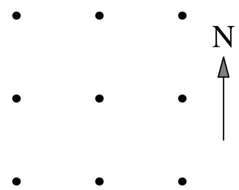
so that $2r = 40$ and $r = 20$.

Commentary

The equal tangent theorem states that the two tangents from a point outside a circle to the circle are equal in length. Here we use two of the vertices of the triangle as the points.

Hence the radius of the circle is 20 cm.

6. Nine dots are arranged in the 2×2 square grid shown. The arrow points north.



Harry and Victoria take it in turns to draw a unit line segment to join two dots in the grid.

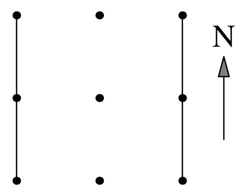
Harry is only allowed to draw an east-west line segment, and Victoria is only allowed to draw a north-south line segment. Harry goes first.

A point is scored when a player draws a line segment that completes a 1×1 square on the grid.

Can either player force a win, no matter how the other person plays?

Solution

Victoria can force a win, by adopting the following strategy.



She plays her first four moves on the sides of the grid, as shown alongside (where only Victoria's moves are shown).

Victoria then plays to complete 1×1 squares.

Why does this strategy work?

If Victoria plays in this way, then Harry is unable to score with his first five moves (since there are no squares with their two north-south edges drawn whenever he plays, so he cannot possibly draw the fourth edge of a square).

After Harry has played for the fifth time, he will have drawn both east-west edges of at least one of the 1×1 squares. Victoria can then draw one of the central two north-south lines to complete at least one square and so score.

When Harry plays his sixth (and final) move, he has no choice where to play. This move may complete a square and win Harry a point (but cannot complete more than one, since one of the central north-south lines remains undrawn so there are two squares which Harry cannot possibly complete yet).

Once Harry has drawn his final segment, there are two adjacent 1×1 squares which share a common undrawn edge, and all other squares have been completed. On her sixth move, therefore, Victoria can draw the final remaining undrawn line segment, completing these final two squares.

Since Harry has scored at most one point and Victoria has scored on two separate occasions, her points total is higher than his, and so she wins.

Note

There are two possible ways to interpret the scoring, depending on whether one or two points are awarded for completing *two* 1×1 squares in the same move. The strategy we give works in either case.

Maclaurin

1. A train leaves K for L at 09:30 while another train leaves L for K at 10:00. The first train arrives in L 40 minutes after the trains pass each other. The second train arrives in K 1 hour and 40 minutes after the trains pass.

Each train travels at a constant speed.

At what time did the trains pass each other?

Solution

Suppose that the first train covers a distance u in one minute and the second train a distance v in one minute, and that the trains pass T minutes after 09:30. Call the crossing point X.

The first train takes T minutes to go from K to X, and the second train takes 100 minutes to travel from X to K. Hence, equating distances, $Tu = 100v$.

The first train takes 40 minutes to travel from X to L, and the second train takes $T - 30$ minutes to travel from L to X, since it started 30 minutes later than the first train. Hence, again equating distances, $40u = (T - 30)v$.

Each of these two equations allows us to find $\frac{u}{v}$. Equating the resulting expressions, we obtain a quadratic equation for T.

We have $\frac{u}{v} = \frac{T-30}{40} = \frac{100}{T}$.

Rearranging, we get $T^2 - 30T - 4000 = 0$, and factorising, we obtain $(T - 80)(T + 50) = 0$ so that $T = 80$ (ignoring the negative answer, which is not practicable here).

Hence the trains pass at 10:50.

2. A right-angled triangle has area $150\,\text{cm}^2$ and the length of its perimeter is 60 cm.

What are the lengths of its sides?

Solution

Let the sides of the triangle adjacent to the right angle be a cm and b cm. Since the area is $150\,\text{cm}^2$ we have $300 = ab$.

The length of the hypotenuse, in cm, is $60 - (a + b)$, so that, using Pythagoras' theorem, we have

$$a^2 + b^2 = 3600 - 120(a + b) + (a + b)^2.$$

After expanding and simplifying, we get

$$120(a + b) = 3600 + 2ab$$

and so, substituting $300 = ab$, we obtain $a + b = 35$. It follows that a and b are roots of the equation $x^2 - 35x + 300 = 0$, which we may factorise as $(x - 20)(x - 15) = 0$. Therefore a and b are 15 and 20, in either order.

Hence the sides of the triangle have lengths 15 cm, 20 cm and 25 cm.

3. Two numbers are such that the sum of their reciprocals is equal to 1. Each of these numbers is then reduced by 1 to give two new numbers.

Prove that these two new numbers are reciprocals of each other.

[*The* reciprocal *of a non-zero number x is the number* $\frac{1}{x}$.]

SOLUTION

Let the original numbers be x and y. We have

$$\frac{1}{x} + \frac{1}{y} = 1. \qquad (*)$$

Multiplying both sides by xy, we obtain

$$y + x = xy.$$

Rearranging, we have

$$1 = xy - x - y + 1,$$

so that

$$1 = (x - 1)(y - 1).$$

Since their product is 1 neither $x - 1$ nor $y - 1$ is 0. Therefore

$$\frac{1}{y-1} = x - 1 \quad \text{and} \quad \frac{1}{x-1} = y - 1,$$

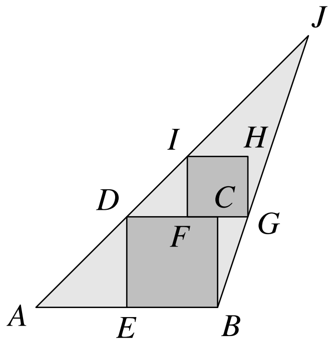
and so the numbers reduced by 1 are reciprocals of each other.

NOTE

It is essential that the argument here is presented in the order shown, going from what we are given to what we have to prove.

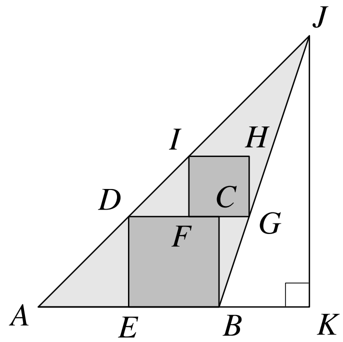
4. The diagram shows the two squares $BCDE$ and $FGHI$ inside the triangle ABJ, where E is the midpoint of AB and C is the midpoint of FG.

What is the ratio of the area of the square $BCDE$ to the area of the triangle ABJ?



Solution

Let K be the foot of the perpendicular from J to AB produced, as shown alongside.



Since E is the midpoint of AB, we have $AE = EB = ED$ and so $\angle DAE = \angle JAK = 45°$.

Let $FG = 2y$ and $BK = x$. Then $FC = CG = y$, $DF = FI = 2y$ and $DC = BC = 3y$.

Triangles GCB and BKJ are similar, since their angles are equal.

Hence $\dfrac{JK}{KB} = \dfrac{CB}{CG} = 3$ and so $JK = 3x$.

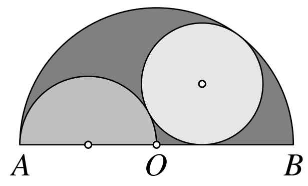
But $JK = AK = 6y + x$. It follows that $3x = 6y + x$ and so $x = 3y$.

The area of $BCDE$ is $9y^2$ and that of ABJ is $27y^2$, so the required ratio is $1 : 3$.

5. A semicircle of radius 1 is drawn inside a semicircle of radius 2, as shown in the diagram, where $OA = OB = 2$.

A circle is drawn so that it touches each of the semicircles and their common diameter, as shown.

What is the radius of the circle?



Solution

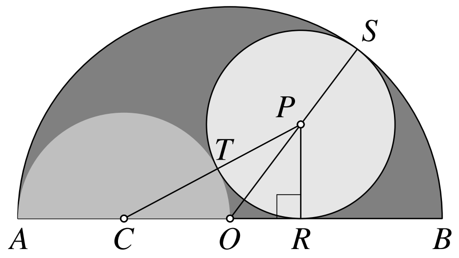
Commentary

A tangent to a circle is perpendicular to the radius through the point of contact; in particular, it follows that the line of centres of two tangent circles passes through the point of tangency. We make use of these facts several times in what follows; for example, when we mention collinearity.

Let the centre of the small semicircle be C, that of the circle be P, and let the circle touch the small semicircle at T, noting that C, T and P are collinear.

Let the circle touch the large semicircle at S. Then O, P and S are also collinear.

Finally let the circle touch the diameter of the large semicircle at R, so that $\angle PRC = 90°$, as shown.



We have $OA = 2$ and $OC = 1$. Let the radius of the circle be r, so that $PO = 2 - r$, and let $OR = x$.

Now, applying Pythagoras' theorem to triangle ORP, we get $x^2 + r^2 = (2 - r)^2$. It follows that

$$x^2 = 4 - 4r. \qquad (1)$$

Also, applying Pythagoras' theorem to triangle CRP, we have $(1 + x)^2 + r^2 = (1 + r)^2$. It follows that

$$2x + x^2 = 2r. \qquad (2)$$

Hence, from equations (1) and (2), $x^2 = 4 - 2(2x + x^2)$ and so $3x^2 + 4x - 4 = 0$, that is, $(3x - 2)(x + 2) = 0$. Taking the positive root (the negative root being impracticable here), we have $x = \frac{2}{3}$ and so $r = \frac{8}{9}$.

Thus the radius of the circle is $\frac{8}{9}$.

6. A tiling of an $n \times n$ square grid is formed using 4×1 tiles.

What are the possible values of n?

[A tiling *has no gaps or overlaps, and no tile goes outside the region being tiled.*]

Solution

Each tile is of area 4, so the area of the grid is a multiple of 4. Hence n is even. We consider two cases, determined by whether n is a multiple of 4 or not.

$n = 4k$, for some positive integer k

In this case the grid can be tiled using $4k^2$ tiles, with each row containing k tiles. Figure 1 shows the case $k = 3$.

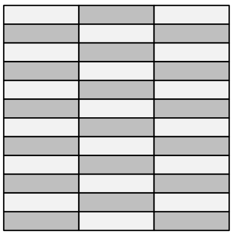


Figure 1

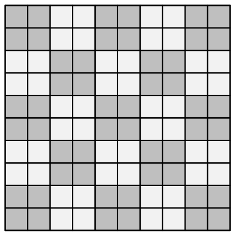


Figure 2

$n = 4k - 2$, for some positive integer k

In this case the grid cannot be tiled. To see this, colour the grid in a 2×2 chessboard pattern, as shown in Figure 2, where $k = 3$.

Notice that, in whatever way a 4×1 tile is placed on the coloured grid, it will cover two cells of each colour. So if it were possible to tile the grid using 4×1 tiles, then they would cover an equal number of 1×1 squares of each colour.

However, there is an odd number of 2×2 squares in the grid, so that the number of 1×1 squares of one colour in the coloured grid is always four more than the number of 1×1 squares of the other colour.

Hence such a tiling is impossible.

Therefore the values of n for which it is possible to tile an $n \times n$ grid with 4×1 tiles are the multiples of 4 (only).