PINK 'KANGAROO' MATHEMATICAL CHALLENGE

Thursday 15th March 2018

Organised by the United Kingdom Mathematics Trust

SOLUTIONS

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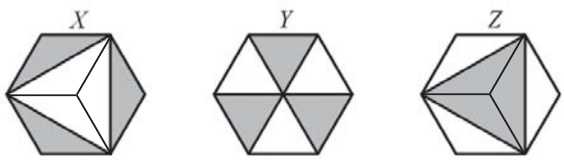
1. **C** In order for the sides to be able to join up, the sum of the lengths of any two sides must be greater than the length of the remaining side (and correspondingly, the difference between the sides must be less than the length of the third side). Hence the third side is less than $5 + 2 = 7$ and more than $5 - 2 = 3$. Therefore it is 5 cm.

2. **C** Let t be the height of the table, b the height of the bottle and c the height of the can (all measured in cm). The first diagram shows $t + b = c + 150$; the second diagram shows $t + c = b + 110$. Adding these equations gives $2t + b + c = 260 + b + c$ so $2t = 260$. So the table has height 130 cm.

3. **E** Let n be the middle number. Then the five numbers are $n - 2, n - 1, n, n + 1, n + 2$ and have sum $5n = 10^{2018}$.

Therefore $n = \dfrac{10^{2018}}{5} = \dfrac{10 \times 10^{2017}}{5} = 2 \times 10^{2017}$.

4. **A** By joining the vertices of the inner triangle to the centre of the hexagon in both the first and third diagrams, it can be seen that each hexagon has been dissected into six equal parts, three of which are shaded. Therefore, X, Y and Z are each half of the hexagon and hence they are all the same.



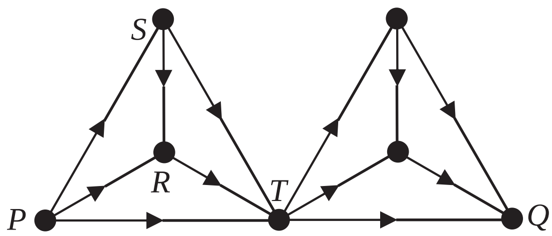
5. **B** If the fruit in each pile is to be identical, then the number of piles must be a factor of 42, 60 and 90. The highest common factor of these numbers is 6, so this is the largest number of piles possible.

6. **B** From the Units digit we see that $5 + S$ ends in a 4, so $S = 9$ (and 1 is carried). Then the Tens digit has $4 + R + 1 = 5$ so $R = 0$ (and nothing carried). Finally, the Hundreds digit gives $P + Q = 6$. Then $P + Q + R + S = 6 + 0 + 9 = 15$.

7. **A** 2018% of 25 is the same as 25% of 2018 (because they are both equal to $\dfrac{25}{100} \times 2018$), so their sum is equal to 50% of 2018, which is 1009.

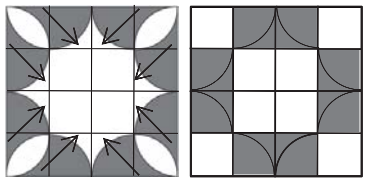
8. **D** All distances in this solution are in metres. Let x be the distance of the bus stop from the second building. Then the total distance walked by the 150 students in this building is $150x$. The distance travelled by each of the students in the first building is $250 - x$, so the total of their distances is $100(250 - x)$. Adding these to get the total distance of all students gives $150x + 100(250 - x) = 50x + 25000$. This is minimised by setting $x = 0$, so the bus stop should be in front of the second building!

9. **B** Label the vertices R, S, T as shown. There are four ways to get from P to T: PT, PRT, PST and $PSRT$. Similarly there are four ways to get from T to Q, so $4 \times 4 = 16$ ways to get from P to Q.



10. **D** The number of terms in the sequence up to and including the n occurrences of n is $1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n+1)$, which equals 105 when $n = 14$. Hence the numbers divisible by 3 are 3 (three times), 6 (six times), 9 (nine times) and 12 (twelve times), giving a total of $3 + 6 + 9 + 12 = 30$ numbers.

11. **C** In the diagram the square of side-length 4 has been dissected into squares of side-length 1. The small curved pieces have then been moved as indicated by the arrows. The resulting shaded area consists of eight squares each with area 1, and the non-shaded area is the same.



12. **E** Each of the trains leaves one town and arrives at another town so there are a total of 80 start/finish points. Forty are already mentioned, so forty must begin/end at Jena.

13. **B** English is taken by 35% of language students, so 65% of language students don't take English. These are stated as 13% of the overall university population. So 1% of the overall university population is the same as 5% of the language students. Hence 20% of the university population is 100% of the language students.

14. **A** Let D be the amount that Peter's dad gave, and B the amount his elder brother gave.
Then $D = \frac{1}{2}(B + 10) \ldots$ (1) and $B = \frac{1}{3}(D + 10) \ldots$ (2).
Substitute (1) into (2) to get $B = \frac{1}{3}\left(\frac{1}{2}(B + 10) + 10\right) = \frac{1}{6}B + 5$.
Hence $\frac{5}{6}B = 5$, so $B = 6$ and $D = \frac{1}{2}(6 + 10) = 8$. This gives a total of $6 + 8 + 10 = 24$. So the book cost 24 euros.

15. **D** Let the three digits be a, b, c. Then the value of the 3-digit number is $100a + 10b + c$ and the 2-digit number (made by deleting the middle digit) is $10a + c$.
Hence $100a + 10b + c = 9(10a + c)$, which gives $10(a + b) = 8c$, or $5(a + b) = 4c$. Hence c is divisible by 5. But c cannot be zero (because this would mean $a + b = 0$, and the digits cannot all be zero). So $c = 5$. Then $a + b = 4$, giving four possibilities: $a = 1, b = 3$ or $a = 2, b = 2$ or $a = 3, b = 1$ or $a = 4, b = 0$. These all give valid solutions ($135 = 9 \times 15,\ 225 = 9 \times 25,\ 315 = 9 \times 35,\ 405 = 9 \times 45$).

16. **E** Square both sides to get $2018^2 + 2018^2 + \ldots + 2018^2 = 2018^{20}$.
Let n be the number of occurrences of 2018^2. Then we get $2018^2 \times n = 2018^{20}$.
So $n = \dfrac{2018^{20}}{2018^2} = 2018^{18}$.

17. **B** Let the integers be $\{2018, a_1, a_2, \ldots, a_n\}$. Their product is $2018 \times a_1 \times a_2 \times \ldots \times a_n = 2018$ so we get $a_1 \times a_2 \times \ldots \times a_n = 1$, meaning each a_i is either 1 or -1 (with an even number of occurrences of -1).

Their sum is $2018 + a_1 + a_2 + \ldots + a_n = 2018$ so $a_1 + a_2 + \ldots + a_n = 0$, meaning an equal number of occurrences of 1 and -1. Hence the list includes 2018 once, an even number of 1s and the same even number of -1s. Therefore the number of integers must be one more than a multiple of 4, so from the list of options only 2017 is possible.

18. **A** In between a vertex labelled n and a vertex labelled m (with $m > n$), there are $m - n + 1$ vertices (including the end points). Hence between vertex 18 and vertex 1018, there are 1001 vertices; between vertex 1018 and vertex 2000 there are 983 vertices. Continuing clockwise from vertex 2000 to vertex 18, we count 37 vertices (and the polygon also has vertex 1018, making 38). Therefore the resulting polygons have 1001, 983 and 38 vertices.

19. **C** Let the four numbers be a, b, c, d with $a \leqslant b \leqslant c \leqslant d$. Abdul chooses the numbers in turn, giving

$$a + \tfrac{1}{3}(b + c + d) = 17 \quad (1) \qquad b + \tfrac{1}{3}(a + c + d) = 21 \quad (2)$$
$$c + \tfrac{1}{3}(a + b + d) = 23 \quad (3) \qquad d + \tfrac{1}{3}(a + b + c) = 29 \quad (4)$$

Adding these equations gives:

$$a + b + c + d + \tfrac{1}{3}(3a + 3b + 3c + 3d) = 90$$

that is

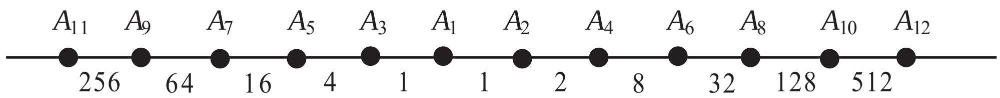
$$2a + 2b + 2c + 2d = 90.$$

Hence $a + b + c + d = 45$. Subtracting a third of this from equation (4), we get

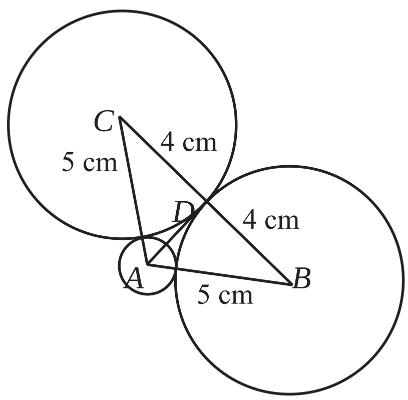
$$d + \tfrac{1}{3}(a + b + c) - \tfrac{1}{3}(a + b + c + d) = 29 - 15.$$

Which gives $\frac{2}{3}d = 14$, so $d = 21$. We can continue in a similar way to find $a = 3$, $b = 9$ and $c = 12$.

20. **E** Let the points O, P etc, be labelled as A_1, A_2, etc, in the order in which they are drawn. Without loss of generality, we can draw the line horizontally with A_2 on the right of A_1. Since A_1 is the midpoint of A_2 and A_3, we then put A_3 to the left of A_1 at a distance of 1 from A_1. Since A_2 is the midpoint of A_3 and A_4, we put A_4 on the right of A_2 at a distance of 2 from A_2. Proceeding in this fashion, the odd points end up further to the left of A_1, and the even points further to the right, with the distances doubling at each stage. The diagram shows the points and the distances between neighbouring points, though not to scale! The distance A_1A_{12} is $1 + 2 + 8 + 32 + 128 + 512 = 683$.



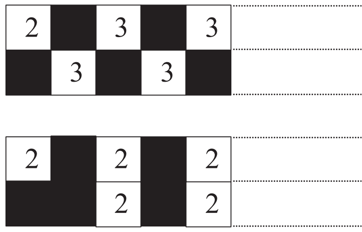
21. **B** The diameter of the non-overlapping circles is the difference between the radius of the large circle and the radius of the small circle, namely $9 - 1 = 8$. Thus the radius of each circle is 4 cm, and the centre of each circle is $1 + 4 = 5$ cm from the centre of the inner circle. The closest the circles could be is when they are touching. The diagram shows two of these circles and the inner circle of radius 1 cm. We need to establish the size of angle CAB. Since $5^2 + 5^2 < 8^2$, we deduce that $\angle CAB > 90°$, and hence it is not possible to fit 4 circles in the annulus.



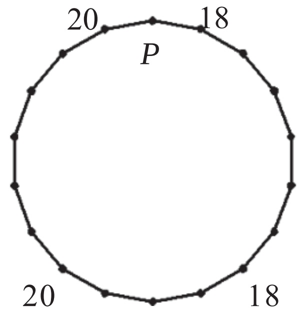
We can also show that $\angle CAB < 120°$. For if $\angle CAB = 120°$, then $\angle CAD = 60°$ and $AD = 2\frac{1}{2}$.

But $AD = \sqrt{5^2 - 4^2} = 3$, so $\angle CAD < 60°$ and $\angle CAB < 120°$. Hence three circles can fit in the annulus, and that is the maximum.

22. D Each black square (except at the endpoints) has at most 3 neighbours so can add a maximum of 3 to the total. This maximum is achieved with the top configuration shown. This gives an average of 3 per column (excluding endpoints). It is possible for a column to have a higher total than three by having two twos (as in the bottom diagram). However, this is at the expense of a total of zero from its adjacent columns, giving a column average of only 2. Hence the maximum is achieved in the first diagram, giving a total of $1007 \times 3 + 2 \times 2 = 3025$.



23. B Let a and b be two adjacent numbers on the vertices. Then the next vertex after b will have the number $b - a$; and that will be followed, in turn, by $-a$, $-b$, $a - b$, then a, b again. This shows that each number reappears every sixth vertex. Therefore the number at the vertex to the left of P is 20 and that to its right is 18. Thus the number at P is $20 + 18 = 38$.



24. D Each column will contain two numbers that add to a multiple of 3. The only possible pair for 3 is 6 so one of these will appear in the top row, and the other on the bottom row. The numbers in the same row as 3 must also add to a multiple of 3, so could be 1, 2 or 1, 5 or 2, 4 or 4, 5. Once this row is decided, the other row is entirely determined by this choice; e.g. if the top row is 3, 1, 2, then the bottom row must be 6, 5, 4. Hence, ignoring order in the first instance, there are just these four choices for what goes in the rows.

3	1	2
6	5	4

3	1	5
6	2	4

3	2	4
6	1	5

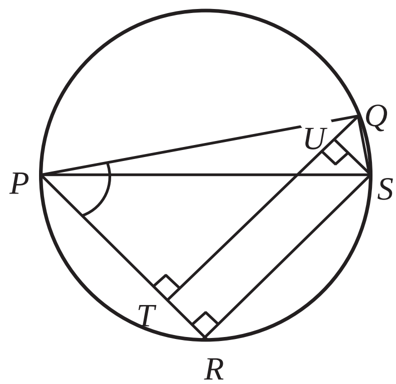
3	4	5
6	2	1

For each of these, the rows can be swapped, which doubles the number of options to 8. For each of these, the three columns can be arranged in 6 ways (3 choices for first column, 2 choices for second, so $3 \times 2 = 6$ choices). Hence there are $8 \times 6 = 48$ choices.

25. D PS is a diameter, so $\angle PRS = 90°$ (angle in a semicircle). Let U be the point on QT for which SU is perpendicular to QT. Hence $RSUT$ is a rectangle, and $SU = TR = 3$.

In triangle TPQ, we have $\angle PTQ = 90°$ and $\angle QPT = 60°$, so $\angle TQP = 30°$. Also, $\angle PQS = 90°$ (angle in a semicircle) so $\angle UQS = 90° - \angle TQP = 90° - 30° = 60°$.

Also $\sin \angle UQS = \dfrac{US}{QS} = \dfrac{3}{QS}$. So



$$QS = \frac{3}{\sin \angle UQS} = \frac{3}{\frac{1}{2}\sqrt{3}} = 3 \times \frac{2}{\sqrt{3}} = 2\sqrt{3}.$$