**SENIOR ‘KANGAROO’ MATHEMATICAL CHALLENGE**

**Friday 1st December 2017**

**Organised by the United Kingdom Mathematics Trust**

*The Senior Kangaroo paper allows students in the UK to test themselves on questions set for the best school-aged mathematicians from across Europe and beyond.*

RULES AND GUIDELINES (to be read before starting):

1. Do not open the paper until the Invigilator tells you to do so.
2. Time allowed: **1 hour**.
3. The use of rough paper is allowed; **calculators** and measuring instruments are **forbidden**.
4. **Use B or HB pencil only** to complete your personal details and record your answers on the machine-readable Answer Sheet provided. **All answers are written using three digits, from 000 to 999.** For example, if you think the answer to a question is 42, write 042 at the top of the answer grid and then code your answer by putting solid black pencil lines through the 0, the 4 and the 2 beneath.

   Please note that the machine that reads your Answer Sheet will only see the solid black lines through the numbers beneath, not the written digits above. You must ensure that you code your answers or you will not receive any marks. There are further instructions and examples on the Answer Sheet.
5. The paper contains 20 questions. Five marks will be awarded for each correct answer. There is no penalty for giving an incorrect answer.
6. The questions on this paper challenge you **to think**, not to guess. Though you will not lose marks for getting answers wrong, you will undoubtedly get more marks, and more satisfaction, by doing a few questions carefully than by guessing lots of answers.

*Enquiries about the Senior Kangaroo should be sent to:*

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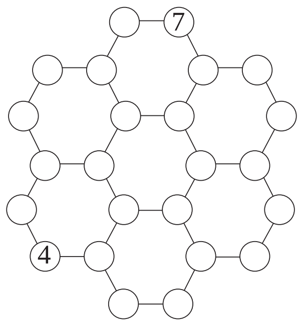
1. An integer is to be written in each circle of the network shown. The integers must be written so that the sum of the numbers at the end of each line segment is the same. Two of the integers have already been written. What is the total of all the integers in the completed diagram?




2. Three sportsmen called Primus, Secundus and Tertius take part in a race every day. Primus wears the number ‘1’ on his shirt, Secundus wears ‘2’ and Tertius wears ‘3’.

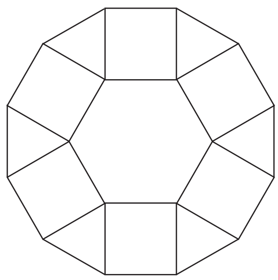
   On Saturday Primus wins, Secundus is second and Tertius is third. Using their shirt numbers this result is recorded as ‘123’.

   On Sunday Primus starts the race in the lead with Secundus in second. During Sunday's race Primus and Secundus change places exactly 9 times, Secundus and Tertius change places exactly 10 times while Primus and Tertius change places exactly 11 times.

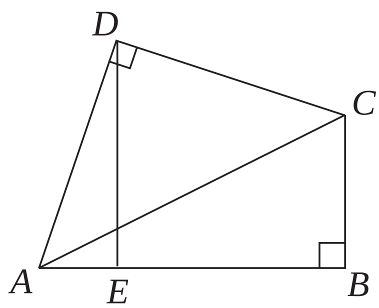
   How will Sunday's result be recorded?

3. All three-digit positive integers whose digit sum is 5 are listed in ascending order. What is the median of this list?

4. The figure shows a shape consisting of a regular hexagon of side 18 cm, six triangles and six squares. The outer perimeter of the shape is $P$ cm. What is the value of $P$?



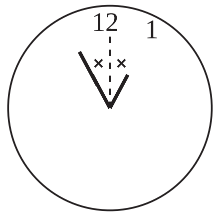
5. The figure shows a quadrilateral $ABCD$ in which $AD = DC$ and $\angle ADC = \angle ABC = 90°$. The point $E$ is the foot of the perpendicular from $D$ to $AB$. The length $DE$ is 25. What is the area of quadrilateral $ABCD$?



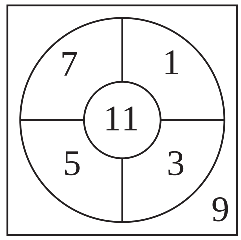

6. Winnie wrote all the integers from 1 to 2017 inclusive on a board. She then erased all the integers that are a multiple of 3. Next she reinstated all those integers that are a multiple of 6. Finally she erased all integers then on the board which are a multiple of 27. Of the 2017 integers that began in the list, how many are now missing?

7. Three rectangles are placed mutually adjacent and without gaps or overlaps to form a larger rectangle. One of the three rectangles has dimensions 70 by 110. Another of the rectangles has dimensions 40 by 80. What is the maximum perimeter of the third rectangle?

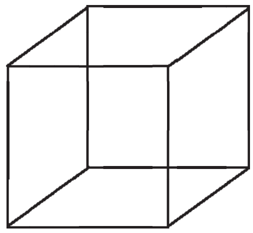
8. Priti is learning a new language called Tedio. During her one hour lesson, which started at midday, she looks at the clock and notices that the hour hand and the minute hand make exactly the same angle with the vertical, as shown in the diagram. How many whole seconds remain until the end of the lesson?




9. Robin shoots three arrows at a target. He earns points for each shot as shown in the figure. However, if any of his arrows miss the target or if any two of his arrows hit adjacent regions of the target, he scores a total of zero. How many different scores can he obtain?

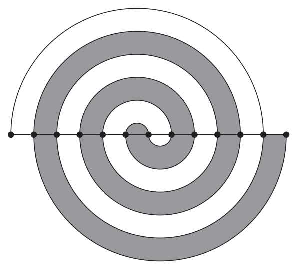



10. At each of the vertices of a cube sits a Bunchkin. Two Bunchkins are said to be adjacent if and only if they sit at either end of one of the cube's edges. Each Bunchkin is either a 'truther', who always tells the truth, or a 'liar', who always lies. All eight Bunchkins say 'I am adjacent to exactly two liars'. What is the maximum number of Bunchkins who are telling the truth?



11. An infinite arithmetic progression of positive integers contains the terms 7, 11, 15, 71, 75 and 79. The first term in the progression is 7. Kim writes down all the possible values of the one-hundredth term in the progression. What is the sum of the numbers Kim writes down?

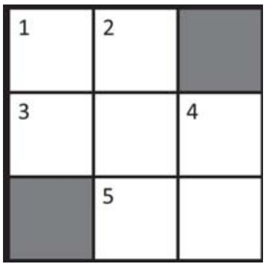
12. The pattern shown in the diagram is constructed using semicircles. Each semicircle has a diameter that lies on the horizontal axis shown and has one of the black dots at either end. The distance between each pair of adjacent black dots is 1 cm. The area, in $cm^2$, of the pattern that is shaded in grey is $\frac{1}{8}k\pi$. What is the value of $k$?



13. In the expression $\frac{k.a.n.g.a.r.o.o}{g.a.m.e}$, different letters stand for different non-zero digits but the same letter always stands for the same digit. What is the smallest possible integer value of the expression?

14. The set $S$ is given by $S = \{1, 2, 3, 4, 5, 6\}$. A non-empty subset $T$ of $S$ has the property that it contains no pair of integers that share a common factor other than 1. How many distinct possibilities are there for $T$?

15. Each square in this cross-number can be filled with a non-zero digit such that all of the conditions in the clues are fulfilled. The digits used are not necessarily distinct.




ACROSS

1. A square
3. The answer to this Kangaroo question
5. A square

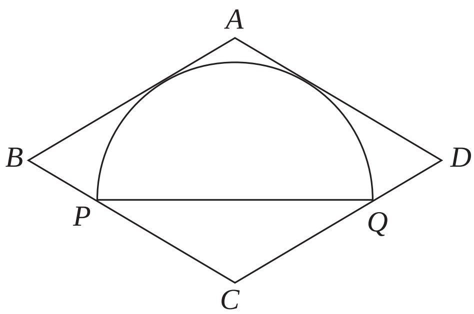
DOWN

1. 4 down minus eleven
2. One less than a cube
4. The highest common factor of 1 down and 4 down is greater than one

16. The curve $x^2 + y^2 = 25$ is drawn. Points on the curve whose $x$-coordinate and $y$-coordinate are both integers are marked with crosses. All of those crosses are joined in turn to create a convex polygon $P$. What is the area of $P$?

17. Matthew writes a list of all three-digit squares backwards. For example, in his list Matthew writes the three-digit square ‘625’ as ‘526’. Norma looks at Matthew's list and notices that some of the numbers are prime numbers. What is the mean of those prime numbers in Matthew's list?

18. The diagram shows a semicircle with diameter $PQ$ inscribed in a rhombus $ABCD$. The rhombus is tangent to the arc of the semicircle in two places. Points $P$ and $Q$ lie on sides $BC$ and $CD$ of the rhombus respectively. The line of symmetry of the semicircle is coincident with the diagonal $AC$ of the rhombus. It is given that $\angle CBA = 60°$. The semicircle has radius 10. The area of the rhombus can be written in the form $a\sqrt{b}$ where $a$ and $b$ are integers and $b$ is prime. What is the value of $ab + a + b$?




19. The sequence of functions $F_1(x)$, $F_2(x)$, … satisfies the following conditions:

$$F_1(x) = x, \qquad F_{n+1}(x) = \frac{1}{1 - F_n(x)}.$$

The integer $C$ is a three-digit cube such that $F_C(C) = C$.
What is the largest possible value of $C$?

20. Let $a$, $b$ and $c$ be positive integers such that $a^2 = 2b^3 = 3c^5$. What is the minimum possible number of factors of $abc$ (including 1 and $abc$)?

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# SOLUTIONS

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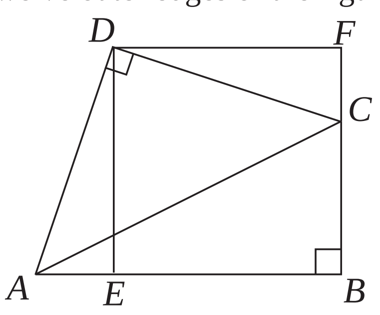
**1.** **132** Each circle immediately adjacent to the initial '4' must contain the same integer, $x$ say, in order for the sum of those numbers at the end of each line segment to be the same. Those circles immediately adjacent to those with the integer $x$ must contain the integer 4 to preserve the sum of those numbers at the end of each line segment. Continuation of this pattern throughout the network eventually yields that the circle marked with a '7' must contain the integer $x$. Therefore $x = 7$. The completed network contains twelve '4's and twelve '7's with a total of $12 \times 4 + 12 \times 7 = 132$.

**2.** **231** At the start of the race, the runners are in order 123. By the end of the race Primus and Tertius have exchanged places 11 times, so Tertius ends the race ahead of Primus. Also, since Secundus and Tertius have exchanged places 10 times, Secundus ends ahead of Tertius. So the result of the race is 231.

**3.** **221** In ascending order the list is 104, 113, 122, 131, 140, 203, 212, 221, 230, 302, 311, 320, 401, 410, 500. The median of this list is 221.

**4.** **216** Each of the squares has a side-length of 18 cm. Therefore each of the triangles has two sides of length 18 cm. Hence the triangles are isosceles. Let the angle contained by the two 18 cm sides of these triangle be $x°$. The interior angles of a square and a regular hexagon are 90° and 120° respectively. By considering angles at a point we have $x + 90 + 90 + 120 = 360$. Therefore $x = 60$ and the triangles are equilateral. All twelve outer edges of the figure are 18 cm in length. Therefore $P = 12 \times 18 = 216$.

**5.** **625**




Draw a line through $D$ that is parallel to $AB$. Let $F$ be the intersection of that line with $BC$ extended, as shown in the diagram.

Now $\angle EDC + \angle CDF = \angle EDC + \angle ADE = 90°$

Therefore $\angle CDF = \angle ADE = 90° - \angle ECD$

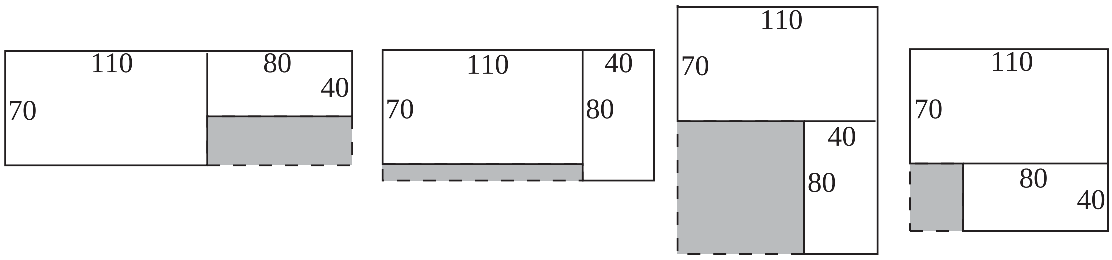
Also $\angle DFC = \angle DEA = 90°$

Therefore triangles $ADE$ and $CDF$ are similar because they have the same set of angles. Because $AD = CD$ they must also be congruent.

Therefore the area of quadrilateral $ABCD$ is equal to the area of rectangle $FDEB$. By the congruent triangles $ADE$ and $CDF$ we know that $DF = DE = 25$. Therefore the required area is $25 \times 25 = 625$.

**6.** **373** Winnie begins with 2017 integers. There are $2016 \div 3 = 672$ multiples of three which are erased. This leaves $2017 - 672 = 1345$ integers. There are $2016 \div 6 = 336$ multiples of six which are reinstated. This leaves $1345 + 336 = 1681$ integers. The only multiples of twenty-seven that then remain in the list are those that are multiples of six. Winnie therefore erases all the multiples of fifty-four. There are $1998 \div 54 = 37$ multiples of fifty-four which are erased. This leaves $1681 - 37 = 1644$ integers. Of the 2017 integers she began with $2017 - 1644 = 373$ are now missing.

**7.** **300**




There are essentially four different configurations as shown in the diagram. The perimeters of the third rectangle in these configurations are 220, 240, 300 and 140 respectively. Therefore the maximum possible perimeter of the third rectangle is 300.

**8.** **276** Let the angle that each hand makes with the vertical be $x$ degrees and let the current time be $s$ seconds after midday.

In one complete hour the hour hand will turn 30°. There are $60 \times 60 = 3600$ seconds in an hour so it takes $3600 \div 30 = 120$ seconds for the hour hand to turn one degree. Therefore $s = 120x$.

In one complete hour the minute hand will turn 360°. There are 3600 seconds in an hour so it takes $3600 \div 360 = 10$ seconds for the minute hand to turn 1°. But the minute hand has turned clockwise through an angle of $(360 - x)°$. Therefore $s = 10(360 - x)$. Equating the two expressions we have obtained for $s$ we obtain the equation $120x = 10(360 - x)$. The solution to this equation is $x = \frac{360}{13}$. Therefore the number of seconds elapsed since midday is $120 \times \frac{360}{13} = \frac{43200}{13} = 3323\frac{1}{13}$. The number of whole seconds remaining is $3600 - 3324 = 276$.

**9.** **013** Robin could score a total of zero either by missing the target with all three arrows or if any two of his arrows hit adjacent regions.

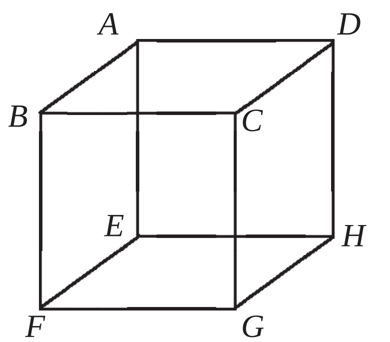
Robin could score totals of 3, 9, 15, 21, 27 or 33 if all three of his arrows hit regions 1, 3, 5, 7, 9 or 11 respectively.

The only scores Robin can obtain from his three arrows hitting non-adjacent regions are $1 + 1 + 5 = 7$, $1 + 5 + 5 = 11$, $3 + 3 + 7 = 13$, $3 + 7 + 7 = 17$, $9 + 9 + 11 = 29$, $9 + 11 + 11 = 31$.

Robin's set of possible score is therefore {0, 3, 7, 9, 11, 13, 15, 17, 21, 27, 29, 31, 33}.

Hence Robin can obtain 13 different possible scores.

**10.** **004** Suppose that there is a truther at $A$. There must be two liars and one truther adjacent to $A$. Let us suppose, without loss of generality, that $B$ is a truther and $D$ and $E$ are liars. Since $B$ is a truther and is adjacent to $A$, then $C$ and $F$ are liars. This shows that there cannot be more than 4 truthers.




If we now suppose that $G$ and $H$ are both truthers, then each of the Bunchkins' statements fits the conditions. So 4 is the maximum possible number of Bunchkins.

**11. 714** The difference between any two terms is either 4 or is a multiple of 4. So the term-to-term difference in the progression must be a divisor of 4. Since all the terms of the progression are integers the only feasible differences are 1, 2 and 4.

If the term-to-term difference is 1 then the one-hundredth term will be $7 + 99 \times 1 = 106$.

If the term-to-term difference is 2 then the one-hundredth term will be $7 + 99 \times 2 = 205$.

If the term-to-term difference is 4 then the one-hundredth term will be $7 + 99 \times 4 = 403$.

The sum of these numbers is 714.

**12. 121** Take each shaded semi-annulus that is below the line and reflect it in the line then move it one centimetre to the left. A shaded semicircle of diameter 11cm is obtained. Therefore the whole shaded area is $\frac{121}{8}\pi$. Hence the value of $k$ is 121.

**13. 002** The expression may be simplified to $\dfrac{k \cdot a \cdot n \cdot r \cdot o \cdot o}{m \cdot e}$.

The smallest possible numerator is $5 \times 4 \times 3 \times 2 \times 1 \times 1 = 120$.

The largest possible denominator is $9 \times 8 = 72$.

The smallest possible value of the expression, whether integer or not, is therefore $\dfrac{120}{72} = 1\dfrac{2}{3}$.

We may obtain a value of 2 for the expression via $\dfrac{6 \times 4 \times 3 \times 2 \times 1}{9 \times 8}$.

Therefore the smallest possible integer value of the expression is 2.

**14. 027** There are six subsets consisting of one element:

{1}, {2}, {3}, {4}, {5}, {6}.

There are eleven subsets consisting of two elements:

{1, 2}, {1,3}, {1,4}, {1, 5}, {1, 6}; {2, 3}, {2, 5}; {3, 4}, {3, 5}: {4, 5}; {5, 6}.

There are eight subsets consisting of three elements:

{1, 2, 3}, {1, 2, 5}; {1, 3, 4}, {1, 3, 5}; {1, 4, 5}; {1, 5, 6}; {2, 3, 5}; {3, 4, 5}.

There are two subsets consisting of four elements:

{1, 2, 3, 5}; {1, 3, 4, 5}.

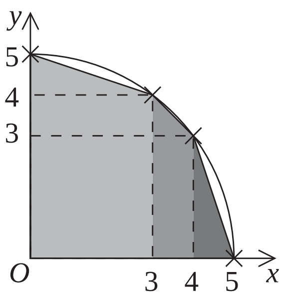
Hence the answer is $6 + 11 + 8 + 2 = 27$.

**15. 829** The possible answers for 1 across are 16, 25, 36, 49, 64 and 81.

The possible answers for 2 down are 124, 215, 342, 511, 728, 999.

By considering the last digit of 1 across (which must be the same as the first digit of 2 down) we see that the only possible pairs of answers for 1 across and 2 down are (25, 511), (49, 999) and (81, 124). The pair (49, 999) would leave no possible answer for 5 across, so may be disregarded. The pair (25, 511) gives an answer of 16 for 5 across and thence 36 for 4 down and 25 for 1 down. However these answers contradict the clue for 4 down so this case may be disregarded. (81, 124) gives an answer of 49 for 5 across and hence 99 for 4 down and 88 for 1 down. These answers satisfy all the conditions in the clues and therefore the answer to this Kangaroo question is 829.

**16. 074** The curve $x^2 + y^2 = 25$ is a circle of radius 5 centred at the origin.




The polygon $P$ has vertices at coordinates (0,5), (3,4), (4,3), (5,0), (4,–3), (3,–4), (0,–5), (–3, –4), (–4, –3), (–5, 0), (–4, 3), (–3, 4).

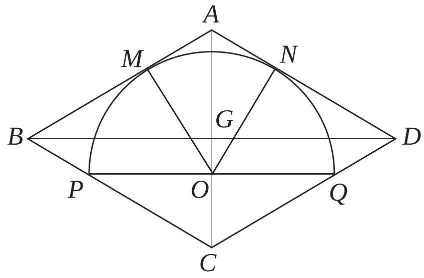
We may find the area of that part of $P$ in the upper-right quadrant by splitting it into two trapezia and a triangle as shown in the diagram. This area is

$$\left(\tfrac{1}{2}(5 + 4) \times 3\right) + \left(\tfrac{1}{2}(4 + 3) \times 1\right) + \left(\tfrac{1}{2} \times 3 \times 1\right) = \tfrac{37}{2}.$$

Therefore the area of $P$ is $4 \times \frac{37}{2} = 74$.

**17. 447** Three-digit squares beginning with 2, 4, 5, 6 or 8 may be disregarded since on reversal these will be divisible by two or five. The residual three-digit squares are 100, 121, 144, 169, 196, 324, 361, 729, 784, 900 and 961. We may disregard 144, 324, 729 and 961 since these are divisible by three and will remain so on reversal. We may also disregard 169, 961, 100 and 121 since these each form a square on reversal. This leaves 196, 361 and 784 which on reversal form 163, 487 and 691. Each of these numbers is prime and they have a mean of $\frac{1}{3}(163 + 487 + 691) = 447$.

**18. 603** Let $O$ be the centre of the semicircle and let $M$ and $N$ be the feet of the perpendiculars drawn from $O$ to $AB$ and $AD$ respectively. Let $G$ be the intersection of the diagonals of the rhombus.




$PO = 10$ and $\angle OPC = 30°$. So $OC = 10\tan 30° = \dfrac{10}{\sqrt{3}}$.

$MO = 10$ and $\angle OAM = 60°$. So $AO = \dfrac{10}{\sin 60°} = \dfrac{20}{\sqrt{3}}$.

Therefore $AC = \dfrac{10}{\sqrt{3}} + \dfrac{20}{\sqrt{3}} = \dfrac{30}{\sqrt{3}} = 10\sqrt{3}$.

Hence $AG = 5\sqrt{3}$ and $\angle GBA = 30°$. So $BG = \dfrac{5\sqrt{3}}{\tan 30°} = 15$.

Therefore the area of triangle $BGA$ is $\frac{1}{2} \times 15 \times 5\sqrt{3} = \frac{75}{2}\sqrt{3}$.

So the area of the rhombus is $4 \times \frac{75}{2}\sqrt{3} = 150\sqrt{3}$.

Therefore $a = 150$ and $b = 3$, so $ab + a + b = 450 + 150 + 3 = 603$.

**19. 343**

$$F_1(x) = x;\ F_2(x) = \frac{1}{1-x};\ F_3(x) = \frac{1}{1 - \frac{1}{1-x}} = 1 - \frac{1}{x};\ F_4(x) = \frac{1}{1 - \left(1 - \frac{1}{x}\right)} = x.$$

Hence we have, $F_1(x) = F_4(x) = F_7(x) = \ldots = F_{3k-2}(x)$.

Therefore we are required to solve $F_{3k-2}(3k - 2) = 3k - 2$ where $3k - 2$ is a three-digit cube (given). The cubes are 125, 216, 343, 512, 729. The only one of the format $3k - 2$ for some positive integer $k$ is 343.

**20. 077** It is clear that each of $a$, $b$ and $c$ must have prime factors including 2 and 3 and, since we are seeking a minimal number of factors of $abc$, these must be the only prime factors.

Let $a = 2^p 3^q$, $b = 2^r 3^s$ and $c = 2^v 3^w$ where $p$, $q$, $r$, $s$, $v$ and $w$ are positive integers or zero. Since $a^2 = 2b^3 = 3c^5$ we have $2^{2p}3^{2q} = 2^{3r+1}3^{3s} = 2^{5v}3^{5w+1}$.

Considering indices of 2 we have $2p = 3r + 1 = 5v$.

The smallest values of $(p, r, v)$ which satisfy this equation are (5, 3, 2).

Considering indices of 3 we have $2q = 3s = 5w + 1$.

The smallest values of $(q, s, w)$ which satisfy this equation are (3, 2, 1).

Using these values, $abc = \left(2^5 3^3\right) \times \left(2^3 3^2\right) \times \left(2^2 3^1\right) = 2^{10}3^6$.

Any factors of $abc$ will be of the form $2^y 3^z$ where $y \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $z \in \{0, 1, 2, 3, 4, 5, 6\}$.

There are 11 possibilities for $y$ and 7 possibilities for $z$. Hence $abc$ has $11 \times 7 = 77$ factors.