SENIOR 'KANGAROO' MATHEMATICAL CHALLENGE

Friday 2nd December 2011

Organised by the United Kingdom Mathematics Trust

The Senior Kangaroo paper allows students in the UK to test themselves on questions set for the best school-aged mathematicians from across Europe and beyond.

RULES AND GUIDELINES (to be read before starting):

1. Do not open the paper until the Invigilator tells you to do so.
2. Time allowed: **1 hour**.
3. The use of rough paper is allowed; **calculators** and measuring instruments are **forbidden**.
4. **Use B or HB pencil only** to complete your personal details and record your answers on the machine-readable Answer Sheet provided. **All answers are written using three digits, from 000 to 999.** For example, if you think the answer to a question is 42, write 042 at the top of the answer grid and then code your answer by putting solid black pencil lines through the 0, the 4 and the 2 beneath.

   Please note that the machine that reads your Answer Sheet will only see the solid black lines through the numbers beneath, not the written digits above. You must ensure that you code your answers or you will not receive any marks. There are further instructions and examples on the Answer Sheet.
5. The paper contains 20 questions. Five marks will be awarded for each correct answer. There is no penalty for giving an incorrect answer.
6. The questions on this paper challenge you **to think**, not to guess. Though you will not lose marks for getting answers wrong, you will undoubtedly get more marks, and more satisfaction, by doing a few questions carefully than by guessing lots of answers.

Enquiries about the Senior Kangaroo should be sent to:

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www.ukmt.org.uk

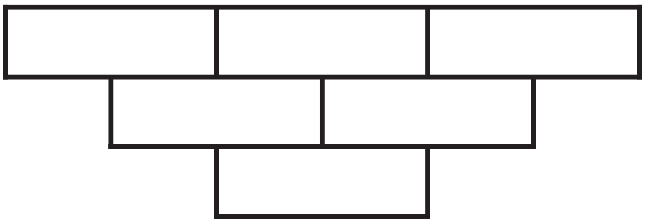
1. The diagram below is to be completed so that:

   a) each cell contains a positive integer;

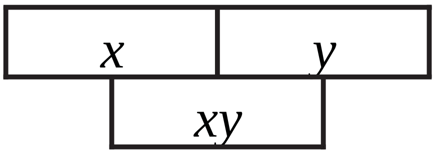
   b) apart from the top row, the number in each cell is the product of the numbers in the two cells immediately above;

   c) the six numbers are all different.

   What is the smallest possible total of the six numbers?



RULE:



2. The mean number of students accepted by a school in the four years 2007 to 2010 was 325. The mean number of students accepted by the school in the five years 2007 to 2011 was 4% higher.

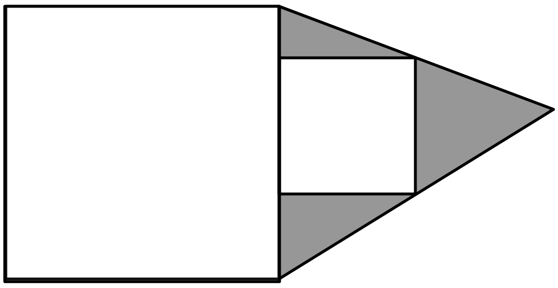
   How many students did this school accept in 2011?

3. 200 people stand in a line. The prize-giver walks along the line 200 times, always starting at the same end. On the first pass, the prize-giver gives each person a pound coin. On the second pass along the line, the prize-giver gives every second person another pound. On the third pass, every third person is given another pound, and so on.

   After 200 passes, how many pounds has the 120th person been given?

4. The diagram below includes two squares: one has sides of length 20 and the other has sides of length 10.

   What is the area of the shaded region?

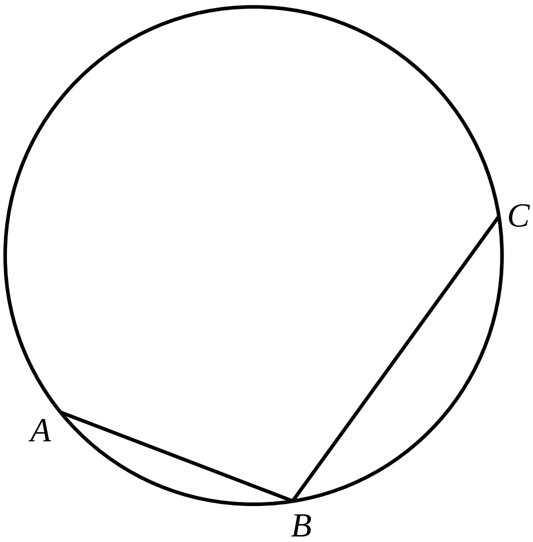


5. How many positive two-digit numbers are there whose square and cube both end in the same digit?

6. The lengths of two sides of an acute-angled triangle and the perpendicular height from the third side of the triangle are 12, 13 and 15 (possibly not in that order).

   What is the area of the triangle?

7. In the diagram, the radius of the circle is equal to the length *AB*.
What is the size of angle *ACB*, in degrees?

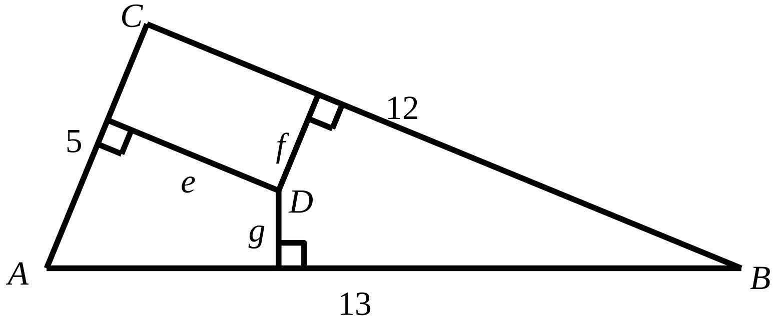


8. The price of an item in pounds and pence is increased by 4%. The new price is exactly n pounds where n is a whole number.
What is the smallest possible value of n?

9. How many squares have $(-1,\ -1)$ as a vertex and at least one of the coordinate axes as an axis of symmetry?

10. What is the value of $\left(\sqrt{8 + 2\sqrt{7}} - \sqrt{8 - 2\sqrt{7}}\right)^2$?

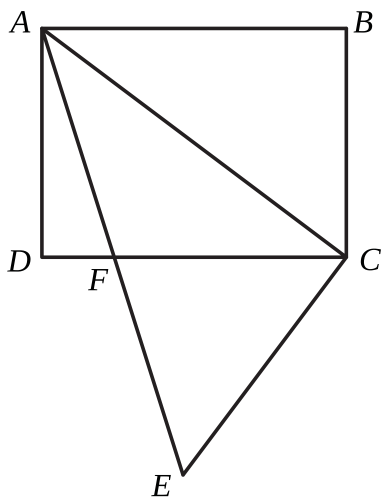
11. In the diagram, *ABC* is a triangle with sides $AB = 13$, $BC = 12$ and $AC = 5$. The point D is any point inside the triangle with $CD = 4$ and the perpendicular distances from D to the sides of the triangle are e, f and g, as shown.
What is the value of $5e + 12f + 13g$?



12. Elections in Herbville were held recently. Everyone who voted for the Broccoli Party had already eaten broccoli. Of those who voted for other parties, 90% had never eaten broccoli. Of those who voted, 46% had eaten broccoli.
What percentage voted for the Broccoli Party?

13. A manager in a store has to determine the price of a sweater. Market research gives him the following data: If the price is €75, then 100 teenagers will buy the sweater. Each time the price is increased by €5, 20 fewer teenagers will buy the sweater. However, each time the price is decreased by €5, 20 sweaters more will be sold. The sweaters cost the company €30 apiece.
What is the sale price that maximizes profits?

14. The diagram shows a rectangle $ABCD$ with $AB = 16$ and $BC = 12$. Angle ACE is a right angle and $CE = 15$. The line segments AE and CD meet at F.
What is the area of triangle ACF?



15. For each real number x, let $f(x)$ be the minimum of the numbers $3x + 1$, $2x + 3$ and $-4x + 24$.
What is the maximum value of $f(x)$?

16. The integer m has ninety-nine digits, all of them nines.
What is the sum of the digits of m^2 ?

17. In rectangle $ABCD$, the midpoints of sides BC, CD and DA are P, Q and R respectively. The point M is the midpoint of QR. The area of triangle APM is a fraction m/n of the area of rectangle $ABCD$, where m and n are integers and m/n is in its simplest form.
What is the value of $m + n$?

18. The integers a, b and c are such that $0 < a < b < c < 10$. The sum of all three-digit numbers that can be formed by a permutation of these three integers is 1554.
What is the value of c?

19. Given that $\left(a + \frac{1}{a}\right)^2 = 6$ and $a^3 + \frac{1}{a^3} = N\sqrt{6}$ and $a > 0$, what is the value of N?

20. The polynomial $f(x)$ is such that $f(x^2 + 1) \equiv x^4 + 4x^2$ and $f(x^2 - 1) \equiv ax^4 + 4bx^2 + c$.
What is the value of $a^2 + b^2 + c^2$?

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SOLUTIONS

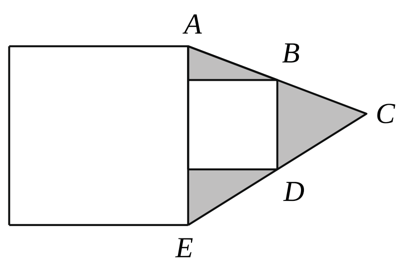
1. **71** Since the entries are different, no entry can be 1, and the smallest total will come from using the integers 2, 3 and 4 on the top line of the diagram. Treating an ordering and its reverse as the same, since they give the same total, we can arrange these in the orders 2, 3, 4 or 2, 4, 3 or 3, 2, 4. Of these, 3, 2, 4 gives the smallest overall total of 71.

2. **390** Over the years 2007 to 2010, the school accepted $325 \times 4 = 1300$ students. The mean for 2007 to 2011 is 4% higher than 325, which is 338. This means that $338 \times 5 = 1690$ students were accepted over the years 2007 to 2011. Therefore 390 students were accepted in 2011.

3. **16** On the first pass, all 200 people receive a pound coin. On the second pass, only people in even numbered positions receive a coin. On the nth pass, people receive a coin if n divides the number representing their position. Now 120 is divisible by 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60 and 120 so the 120th person receives 16 pound coins.

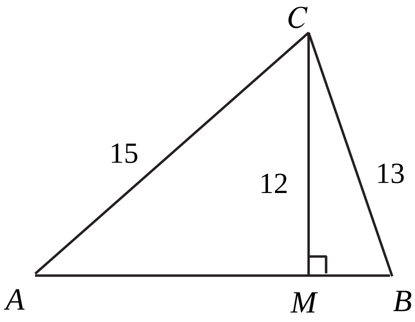
[*Note that, in general, if the prime decomposition of an integer, X, is* $p_1^{a_1}p_2^{a_2}p_3^{a_3}\ldots\ p_n^{a_n}$ *then the number of divisors of X is* $(a_1 + 1)(a_2 + 1)(a_3 + 1)\ldots(a_n + 1)$.]

4. **100** Using the labelling shown, we see that ΔACE and ΔBCD are similar and have lengths in the ratio 2:1. Because the height of ΔACE is 10 + the height of ΔBCD, the height of ΔACE is 20 and its area is $\frac{1}{2} \times 20 \times 20 = 200$. The area of the smaller square is 100 so the shaded area is $200 - 100 = 100$.



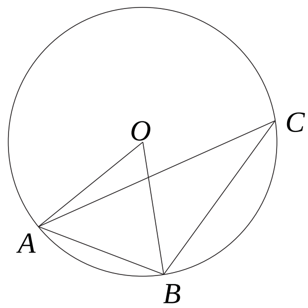
5. **36** The square and cube of an integer end in the same digit if, and only if, the integer itself ends in 0, 1, 5 or 6. The two-digit numbers with this property can have any tens digit from 1 to 9 so there are $4 \times 9 = 36$ such two-digit numbers.

6. **84** The perpendicular height from any side of an acute-angled triangle is always less than the length of either of the other two sides so the height is 12. Thus we have the situation shown in the diagram alongside.



Using Pythagoras' theorem in ΔACM, we obtain $AM = 9$ and in ΔCMB we get $MB = 5$. This means that $AB = 14$ and the area of triangle ABC is $\frac{1}{2} \times 14 \times 12 = 84$.

7. **30** Let the centre of the circle be O. Join A to C and O to A and B, as shown in the diagram alongside.



In ΔAOB, $AO = OB$ since they are both radii, and we are given that AB has length equal to the radius so $AB = AO = OB$ and ΔAOB is equilateral. Hence $\angle AOB = 60°$. Since the angle at the centre is twice the angle on the circumference, $\angle ACB$ is 30°.

8. **13** Let the original price, in pence, be p.

The new price is 4% more than the original so, working in pence,

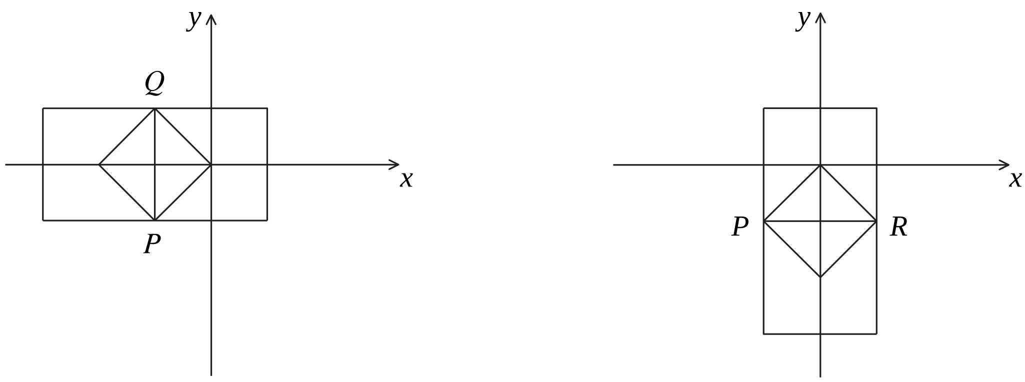
$$100n = \frac{104}{100} \times p,$$

which may be rearranged to

$$n = \frac{13}{1250} \times p.$$

Now n is an integer and 1250 is not divisible by 13, so p is divisible by 1250. The smallest value of n will be when $p = 1250$, which means n is 13.

9. **5**



Let P be $(-1, -1)$ and suppose that the x-axis is a line of symmetry. Then $Q\,(-1, 1)$ is a vertex of the square since it is the reflection of the vertex P in the x-axis. Hence PQ is either an edge or a diagonal of the square. In the first case there are two possible squares and in the second case there is one, as shown in the first figure.

Similarly, when the y-axis is a line of symmetry there are three possible squares. However, one of these is the same as before, so in all there are exactly five squares possible.

10. **4** By expanding the brackets, we obtain

$$\left(\sqrt{8+2\sqrt{7}}-\sqrt{8-2\sqrt{7}}\right)^2 = 8+2\sqrt{7}-2\sqrt{\left(8+2\sqrt{7}\right)\left(8-2\sqrt{7}\right)}+8-2\sqrt{7}$$
$$= 16-2\sqrt{64-28} = 16-2\sqrt{36} = 16-12 = 4.$$

11. **60** Area ΔABC = area ΔACD + area ΔBCD + area ΔABD

$$= \tfrac{1}{2}\times e\times 5 + \tfrac{1}{2}\times f\times 12 + \tfrac{1}{2}\times g\times 13 = \tfrac{1}{2}\left(5e+12f+13g\right).$$

But ΔABC has sides 5, 12 and 13, hence it is a right-angled triangle and so has area $\tfrac{1}{2}\times 5\times 12 = 30$. Therefore $5e+12f+13g = 60$.

12. **40** From the information given about those who voted, we can conclude:

		Eaten broccoli? Yes	Eaten broccoli? No
Voted Broccoli Party?	Yes	y	0
	No	x	$9x$

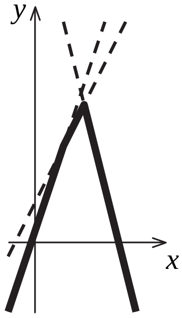
where x, $9x$ and y are the appropriate percentages of those who voted.

We are given that $x+y = 46$ and, since the table includes everyone, we also have $x+y+9x = 100$. So $9x = 54$ and $x = 6$. Therefore $y = 40$ and so the percentage that voted for the Broccoli Party is 40%.

13. **65** Suppose n represents the number of increments of €5 above (or below, if n is negative) the selling price of €75. Then the number of sweaters sold is $100-20n$ and the profit made, in Euros, is $\left((75+5n)-30\right)(100-20n) = (45+5n)(100-20n) = 100(5-n)(9+n)$. So the profit is $100\left(49-(n+2)^2\right)$ and is a maximum when $n = -2$. This gives a sale price of €65.

14. **75** Using Pythagoras' theorem firstly in ΔABC and then in ΔACE we get $AC = 20$ and $AE = 25$. It follows that ΔABC is similar to ΔACE as the corresponding sides are in the same ratio. Therefore, $\angle BAC = \angle CAE$. Also $\angle BAC = \angle ACF$, using alternate angles, so $\angle CAF = \angle ACF$ and ΔAFC is isosceles. Let M be the mid-point of AC and join M to F. This gives two more right-angled triangles, ΔAMF and ΔCMF, also similar to ΔABC. Thus $\frac{MF}{MA} = \frac{BC}{BA}$ which gives $MF = \frac{15}{2}$. Therefore the area of ΔACF is $\frac{1}{2}\times\frac{15}{2}\times 20 = 75$.

15. **10**

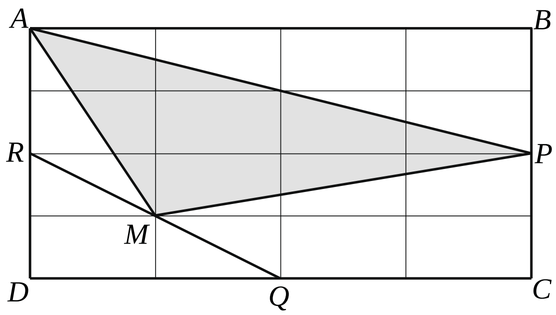


The diagram shows the dashed lines with equations $y = 3x+1$, $y = 2x+3$ and $y = -4x+24$. The solid lines form the graph of the function given in the question. We can see that the maximum value of $f(x)$ occurs when $y = 2x+3$ crosses $y = -4x+24$.

At this point $y = 10$, therefore the maximum value of $f(x)$ is 10.

16. 891 Since $m = 10^{99} - 1$, we have $m^2 = (10^{99} - 1)^2 = 10^{198} - 2 \times 10^{99} + 1$ $= 999\ldots 9998000\ldots 001$ where there are 98 nines and 98 zeroes. Therefore the sum of the digits is $98 \times 9 + 8 + 1 = 891$.

17. 21



Dividing rectangle *ABCD* into 16 equal parts, as shown in the diagram above, demonstrates that the area of $\Delta APM = 12 - \frac{1}{2} \times 3 - \frac{1}{2} \times 3 - \frac{1}{2} \times 8 = 5$ parts. Therefore the area of ΔAPM is $\frac{5}{16}$ of the area of rectangle *ABCD* so $m + n = 21$.

18. 4 There are six different numbers that can be formed with digits a, b and c. The sum of these six numbers is

$$\begin{aligned}&(100a + 10b + c) + (100a + 10c + b) + (100b + 10a + c)\\ &+ (100b + 10c + a) + (100c + 10a + b) + (100c + 10b + a)\\ &= 200(a + b + c) + 20(a + b + c) + 2(a + b + c)\\ &= 222(a + b + c) = 1554\end{aligned}$$

so $a + b + c = 7$. Thus the only possibility for a, b and c is 1, 2 and 4 so $c = 4$.

19. 3 From $\left(a + \frac{1}{a}\right)^2 = 6$ we have $a + \frac{1}{a} = \sqrt{6}$ since $a > 0$. Therefore $\left(a + \frac{1}{a}\right)^3 = (\sqrt{6})^3$, which gives $a^3 + 3a^2 \times \frac{1}{a} + 3a \times \frac{1}{a^2} + \frac{1}{a^3} = 6\sqrt{6}$ and so $N\sqrt{6} + 3\left(a + \frac{1}{a}\right) = 6\sqrt{6}$. This means that $N = 3$.

20. 17 From $f(x^2 + 1) \equiv x^4 + 4x^2 \equiv (x^2 + 1)^2 + 2(x^2 + 1) - 3$ we deduce that $f(w) \equiv w^2 + 2w - 3$ and hence that $f(x^2 - 1) \equiv (x^2 - 1)^2 + 2(x^2 - 1) - 3 \equiv x^4 - 2x^2 + 1 + 2x^2 - 2 - 3 \equiv x^4 - 4$. This means $a = 1$, $b = 0$ and $c = -4$. Therefore the value of $a^2 + b^2 + c^2$ is 17.

An alternative solution is to realise that $f(x^2 + 1) \equiv [(x^2 + 1) + 1]^2 - 4$. So $f(x^2 - 1) \equiv [(x^2 - 1) + 1]^2 - 4 \equiv x^4 - 4$. This gives the same value for $a^2 + b^2 + c^2$.