United Kingdom
Mathematics Trust

Notes of use to UKMT mentees

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(Pythagoras, Hypatia, Archimedes and Mary Cartwright papers)

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Whereas these notes were originally written for largely younger students engaged in working on problems at the level of the Pythagoras, Hypatia and Archimedes papers, students working on Mary Cartwright papers may also find some aspects of these notes useful.

There are UKMT books on the subject of Geometry. They can be found on the UKMT website at `http://shop.ukmt.org.uk/ukmt-books/`. Crossing the Bridge by Gerry Leversha and Plain Euclidean Geometry by A.D.Gardiner and C.J.Bradley tackle this subject in more detail. The former is the more recent publication. These books would be useful for students developing their geometrical skills for taking part in competitions.

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A. Even, odd and prime integers

Integers appear in many mathematical problems. You first met them when you were very young starting to count but integers include both positive and negative numbers as well as zero. Sometimes we want to refer to a subset such as the non-negative integers; by implication this set must include zero.

If a sum or difference of a pair of integers is even, then both integers are even or both are odd. To get an odd sum, one integer must be even and the other odd. If a product of two integers is odd, then both integers are odd. Writing $2k$ where k is an integer is a nice way of writing a general expression for an even number. Similarly $2k + 1$ is a general expression for an odd number.

Note that 2 is a *prime* integer. It is the only even prime integer. Some find this surprising but then 3 is the only prime that is a multiple of 3. All primes apart from 2 are odd. Not every odd number is prime.

The integer 1 is *not* prime. There would be problems if we included it. This is because every positive integer greater than 1 is either prime or can be written as a product of a unique finite list of primes. If we considered 1 as prime, then we could keep on dividing out primes from a number *ad infinitum*. Every positive integer greater than 1 which is not prime is called *composite*.

B. Adding series of numbers

Adding a set of numbers going up in regular steps, for example $3 + 8 + 13 + \cdots + 98$, is easy with the right trick. Think of this written two ways :

$$\begin{array}{c} 3 + \ \ 8 + 13 + \cdots + 88 + 93 + 98 \\ 98 + 93 + 88 + \cdots + 13 + \ \ 8 + \ \ 3 \end{array}$$

You can now see that we have 20 pairs of numbers each adding to 101. (Why are there 20 pairs and not $(98 - 3)/5 = 19$ pairs?) Adding all of these pairs together gives a total of $20 \times 101 = 2020$ so the original series adds up to half of this, namely 1010.

A series of numbers going up in regular steps like this is called *arithmetic*. The word *series* is applied when we are adding a set of numbers. The word used when we are simply listing them is *sequence*.

C. Useful algebraic factorisations

If you have come across multiplying out brackets, for example:

$$(a+b)(c+d) = ac + ad + bc + bd$$

then you may know that to multiply out a squared bracket you should repeat it like this:

$$(a+b)^2 = (a+b)(a+b) \qquad \text{and then expand this product}$$

These results are useful and worth remembering:

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad \text{and} \qquad (a-b)^2 = a^2 - 2ab + b^2$$

A third result which is very useful is:

$$(a+b)(a-b) = a^2 - b^2$$

which is sometimes called (the factorisation of) the difference of two squares.

Since a square number is always non-negative, the second result leads to $a^2 - 2ab + b^2 \geq 0$.

This can be written in various ways, for example:

$$a^2 + b^2 \geq 2ab; \qquad \frac{a^2+b^2}{2} \geq ab; \qquad \frac{a^2+b^2}{2} \geq \sqrt{a^2b^2}$$

If we now replace a^2 with x and b^2 with y, we deduce that:

$$\frac{x+y}{2} \geq \sqrt{xy} \qquad \text{for} \quad x, y \geq 0.$$

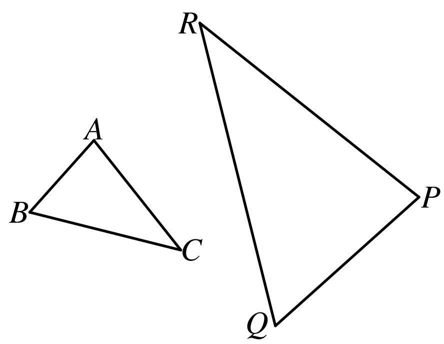
Equality only occurs if and only if $x = y$.

This is known as the arithmetic mean-geometric mean inequality (or "the AM-GM inequality" for short). On the left hand side of $\geq$ is the *arithmetic mean* of x and y which the general public often call the average of x and y but in school is called the mean of x and y. On the right hand side is the *geometric mean* of x and y.

We can use this, for example, to verify that $\sqrt{15} < \frac{1}{2}(3+5)$, or for instance, to prove that the rectangle with the greatest area for a given perimeter is a square.

D. Similar and congruent triangles

Two triangles are called *similar* if both have the same set of angles. The *corresponding* sides are then in a fixed ratio to each other. A side from one triangle corresponds to a side in the other triangle if the two sides are opposite the same angle. (Care might have to be taken with the order of sides if the triangles were isosceles.) Consider these two triangles.



If $\angle ABC = \angle PQR$ and $\angle CAB = \angle RPQ$, by the angle sum of a triangle the remaining pairs of angles must be equal. It follow that $\triangle ABC$ is similar to $\triangle PQR$. We can now write down equal ratios.

$$\frac{AB}{PQ} = \frac{BC}{QR},$$

by comparing the sizes of the two triangles, or

$$\frac{AB}{BC} = \frac{PQ}{QR},$$

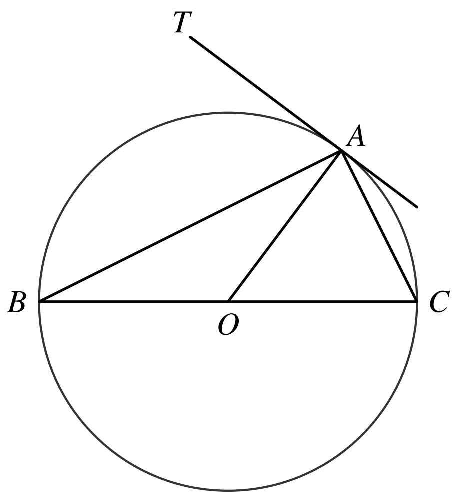
which yields the same calculation but compares the shapes of each triangle.

Triangles are called *congruent* if they have the same set of angles and also their corresponding sides are equal. Congruent triangles can be mapped one to the other by a suitable combination of translations, rotations and reflections. There are four ways you can prove two triangles are congrent. A brief notation that you can use in a proof is given in brackets.

- Two pairs of angles and the sides between the angles are correspondingly equal. (*ASA*)
- Two pairs of sides and the angles between the sides are correspondingly equal. (*SAS*)
- Three pairs of sides are correspondingly equal. (*SSS*)
- A pair of right angles, a pair of hypotenuses and another pair of sides are correspondingly equal. (*RHS*)

In the last, the angle does not have to be between the two sides like *SAS* because the angle is 90°.

E. Right angles in a circle



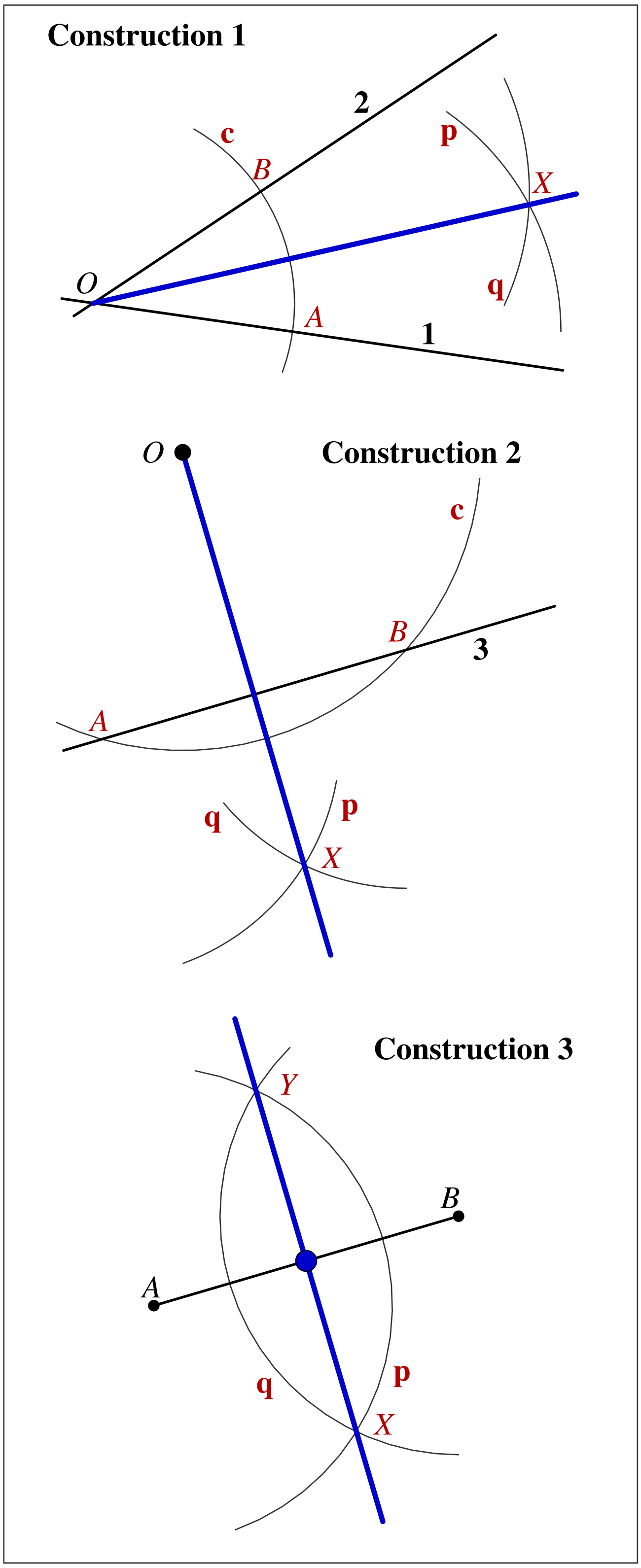
In the diagram A, B and C are on the circle, O is the centre of the circle and also on the line BC. Hence BC is a *diameter* of the circle. It is left to the student to work out why there are two isosceles triangles in the diagram. We can now see that:

$$\angle BAO + \angle OAC = 90^\circ$$

This can be briefly stated as '*angle in semi-circle is a right angle*'. The *converse* is that if $\triangle ABC$ has a right angle at A, then a circle on the hypotenuse BC as diameter will pass through A.

TA is a *tangent* to the circle at A. $\angle OAT = 90^\circ$. This is proved in section H.

For many decades students were expected to be able to carry out constructions using a straight edge and a pair of compasses. For this purpose, a ruler is a straight edge. You may use the measuring marks to set a pair of compasses to a fixed radius but not use them, for example, to work out the midpoint of a line. In the diagrams below points and lines in black are starting positions. Red labels and grey lines indicate constructions. Blue indicates the constructed line or point.



1. Angle bisector

You wish to bisect the angle at O between two lines labelled **1** and **2** which meet at the point O. Start by drawing part of circle **c** of convenient radius with centre O to cut the lines at A and B. This is called an *arc* of the circle. This will work whether the angle is acute or obtuse.
At this stage you may change the radius at which you set your pair of compasses. Draw arc **p** with centre A and then with the *same* radius draw arc **q** with centre B.
X is the intersection of these two arcs.
Line OX is the required angle bisector.

2. Perpendicular line

You wish to draw a line from O at right angles to a line labelled **3**. O may lie on the line. Start by drawing an arc **c** of convenient radius with centre O to cut the line and label intersections A and B.
At this stage you may change the radius at which you set your pair of compasses. Draw arc **p** with centre A and then with the *same* radius draw arc **q** with centre B.
X is the intersection of these two arcs.
Line OX is the required perpendicular.

3. Perpendicular bisector

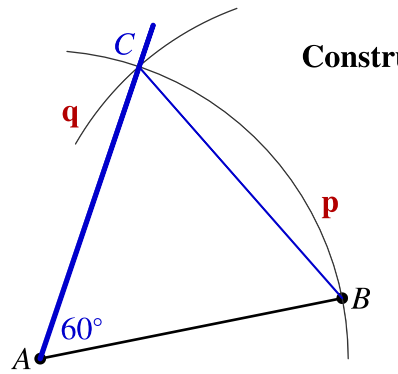
This construction will also find the mid-point of a segment of a line. Start with a segment of a line marked by points A and B. Draw an arc **p** with centre A and radius at least half the length of the segment. With the *same* radius draw arc **q** with centre B. X is one intersection of these two arcs, Y the other intersection. Join XY. It is perpendicular to AB and intersects AB at the mid-point of AB.

4. Angles and triangles

The previous construction also provides a way of forming a *rhombus* $AXBY$ based on one diagonal being AB. We can also:

- Ignore one of the intersections X or Y which provides a means of constructing an *isosceles* triangle.
- Set the radius of the pair of compasses to the length of AB. Draw arc **p** with centre A. With the *same* radius draw arc **q** with centre B to cut this arc. Call this intersection C. Then $\angle BAC = 60^\circ$ and $\triangle ABC$ is equilateral.

Construction 4

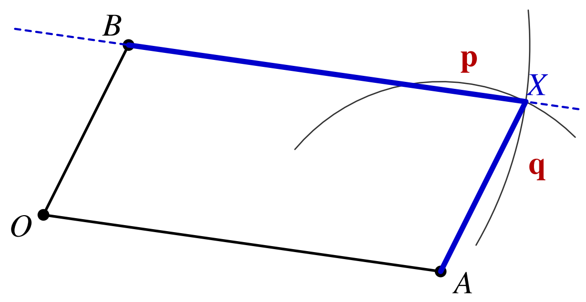


5. Parallel lines

Here are two variations:

- Given three vertices of a parallelogram O, A and B, a variation of the first construction allows us to find a fourth point X so that BX is parallel to OA and AX is parallel to OB. Set the radius of the pair of compasses to the length of OB. Draw arc **p** with centre A. Reset the radius to the length of OA. Draw arc **q** with centre B. Call this intersection X. $OAXB$ is the required parallelogram.
- You will note that this method also allows you to construct just the line through B parallel to OA as indicated by the dotted line.

Construction 5

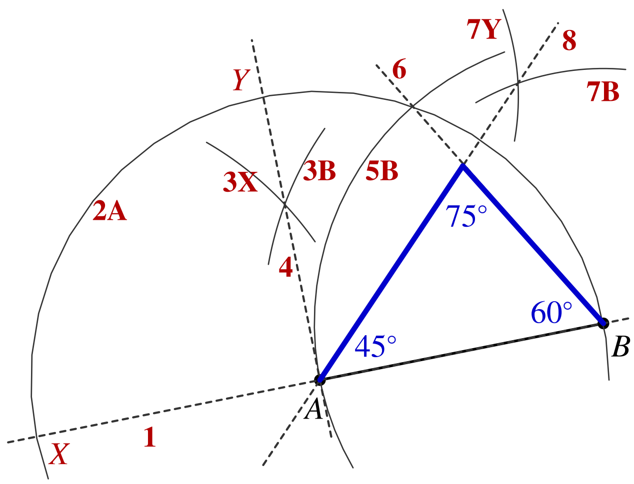


6. Scalene triangles

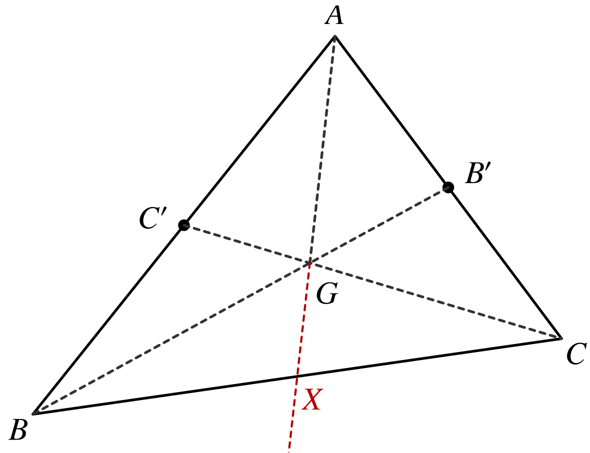
When drawing a triangle in order to prove some general result about a triangle, it is quite difficult to avoid drawing a triangle which looks either isosceles or right-angled. This can be avoided by using angles close to 45°, 60° and 75°. This can usually be done by eye.

This triangle is used as an example to illustrate the methods above. It is based on bisecting a right angle to produce 45°. The construction order is indicated by the numbers in red along with the centre used if an arc is drawn. The radii are are not fixed except where an arc clearly passes through an established point.

Construction 6



G. Centroid of a triangle



Triangles have many interesting properties, including those of their *centres*. This is one centre.

> There are three lines joining the vertices of the triangle to the mid-points of the opposite sides. These are called the *medians* of the triangle. The three medians meet at the *centroid* of the triangle. The centroid divides the length of each median in the ratio 2 : 1. The centroid can be thought of as the 'balancing point' of the triangle. If you cut a triangle out of card, it will balance on a ruler if this point is above the ruler. If the triangle is hung from A, the centroid will lie below A.

Proof:

We write $|ABC|$ to mean the area of $\triangle ABC$. Let B' and C' be the midpoints of CA and AB respectively, and let the medians BB' and CC' meet at G. We draw the line AG and extend it to meet BC at X. We are going to prove that X is the midpoint of BC.

Because the bases AB' and CB' of $\triangle ABB'$ and $\triangle CBB'$ are equal and because these two triangles have the same height measured from B, it follows that:

$$|ABB'| = |CBB'|$$

For the same reasons, $|AB'G| = |CB'G|$, so it follows that $|AGB| = |CBG|$.
Similarly $|ACG| = |BCG|$.
Hence we have now shown that $\triangle ABG$ has the same area as $\triangle ACG$.

Let $BX = k \times CX$. We can show that $|GBX| = k \times |GCX|$ and $|ABX| = k \times |ACX|$. By subtracting areas, we have $|ABG| = k \times |ACG|$.

Thus $k = 1$ and X is the midpoint of BC.

Furthermore:

$$\begin{aligned} \text{because} \quad & |BGX| = |CGX| \\ & |BGX| = \tfrac{1}{2}|CGB| = \tfrac{1}{2}|AGB| \\ & \therefore GX = \tfrac{1}{2} \times AG \end{aligned}$$

because $\triangle BGX$ and $\triangle AGB$ have the same height from B, and bases GX and AG.

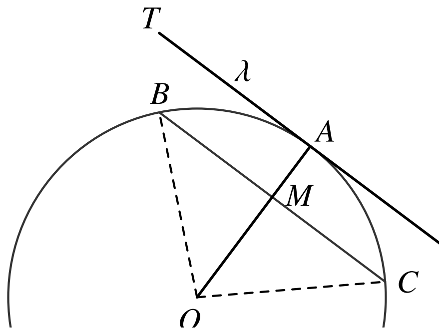
> When considering the area of a triangle, you may find it useful to turn the paper round so that the side you want to treat as the base is closest to you and the height is measured away from you.

H. Circle geometry

The content of this section is more appropriate to papers at Archimedes level and harder.

(a) *Why should a tangent be perpendicular to the radius of the circle to the point of contact?*

We first show that a radius which bisects a chord is perpendicular to the chord.



Let M be the midpoint of a chord BC of a circle with centre O. $OB = OC$ as both are radii of the circle, so $\angle OBC = \angle OCB$. We show that $\triangle OBM$ and $\triangle OCM$ are congruent using the *SSS* rule.

This is because $BM = CM$ and OM is a common side, therefore $\triangle OBM$ is congruent to $\triangle OCM$.

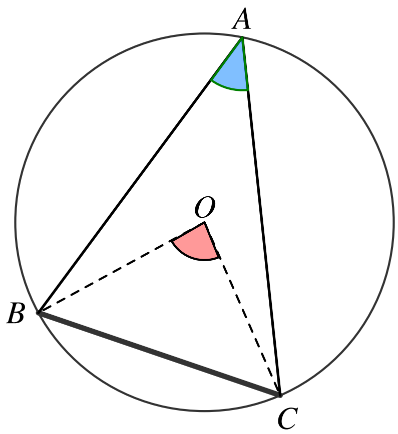
It follows that $\angle OMB = \angle OMC$. Since $\angle OMB + \angle OMC = 180°$, $\angle OMB = \angle OMC = 90°$.
Another way of saying this is that OM is perpendicular to BC.

We could prove the converse: if OM is perpendicular to BC, then M is the midpoint of BC.

Let OM meet the circle at A. Construct the perpendicular to OA through A (and hence parallel to BC). Call it λ. We claim that every point on λ lies outside the circle, except for A itself. Let T be the point where the line through O and B meets λ. We show that $\triangle OAT$ is similar to $\triangle OMB$. This is because $\angle BOA = \angle TOA$ and $\angle OMB = \angle OAT = 90°$. Because the chord lies within the circle, $OM < OA$. Hence $OB < OT$. Hence T lies outside the circle. This is true for every point T on λ other than A. Hence λ is a tangent to the circle.

From the properties of radii, chords and tangents we can deduce a number of significant theorems.

(b) *The angle subtended by a chord at the centre is twice that at the circumference.*

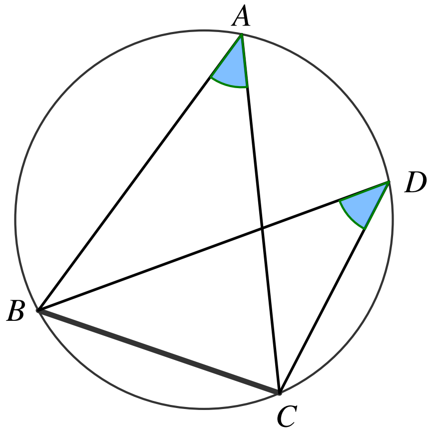


What *subtend* means here is that the chord BC 'pulls apart' an angle at O and another angle at A.

$$\angle BOC = 2 \times \angle BAC$$

This can be proved by drawing in the line OA and considering the isosceles triangles which are formed. Note that there is more than one configuration for the diagram, for instance with AB crossing OC.

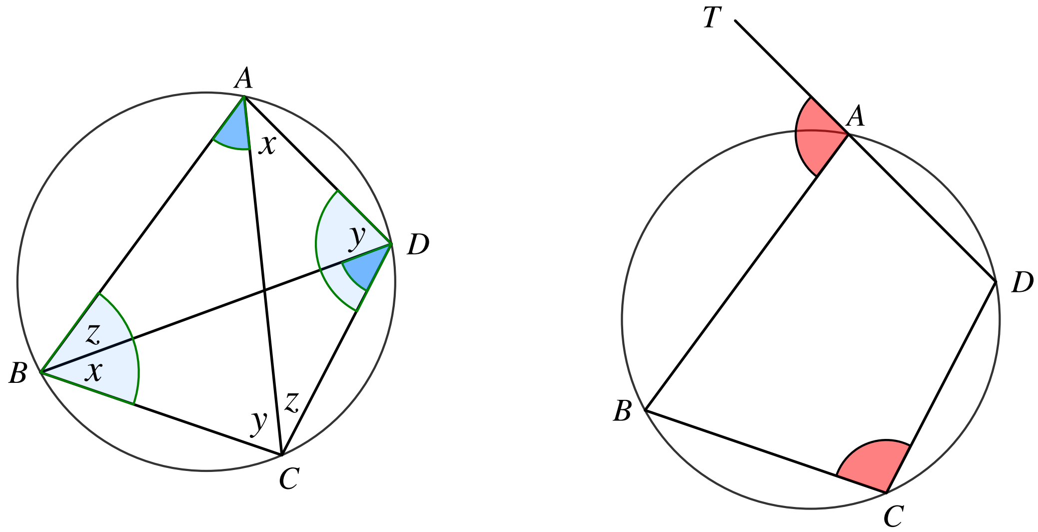
(c) *Angles subtended by a chord at the circumference in the same segment are equal*



This can be proved by using the previous theorem. If O is the centre of the circle (not shown), $\angle BOC = 2 \times \angle BAC$ *and* $\angle BOC = 2 \times \angle BDC$ therefore

$$\angle BAC = \angle BDC.$$

(d) *Opposite angles in a cyclic quadrilateral are supplementary, that is, add to* 180°.



There are various ways of proving this. The angle marks in the left hand figure should suggest a way. Alternately, we could put O back in the figure, delete the diagonals and note that $\angle AOC$ on the side of B and $\angle AOC$ on the side of D sum to 360°. These angles are twice $\angle ABC$ and $\angle ADC$ respectively. We can even go further back to basics and note that joining O to A, B, C and D forms four isosceles triangles. Thus

$$\angle ABC + \angle ADC = 180°.$$

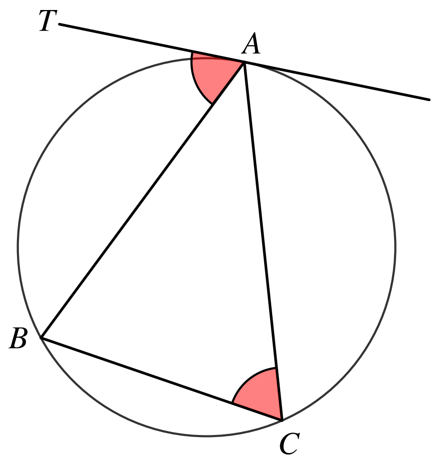
This also applies to the other opposite pair. If we extend DA as in the second figure:

$$\angle BCD = \angle BAT$$

We say:

(e) *The external angle of a cyclic quadrilateral is equal to the interior opposite angle.*

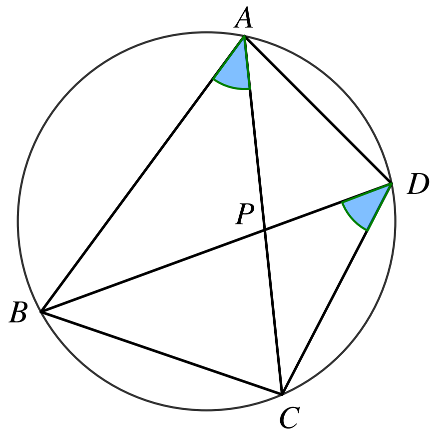
(f) *The alternate segment theorem.*



This is easiest to show in the case that AC is perpendicular to the tangent, for then $\angle ABC$ is a right angle. (See section E on page 4.) If it is true in this case, then it is *always* true, as if A and B are fixed, then $\angle BAT$ is fixed and $\angle ACB$ can not change in value by theorem (c). Alternatively you could join all of the points A, B and C to O and use isosceles triangles.

You might notice that is not unlike the previous diagram except that D has moved to coincide with A. The line DA has then coincided with the tangent at A.

(g) *Some similar triangles*



Reminding ourselves of the pairs of equal angles shown in part (d) above, we can conclude that $\triangle BAP$ is similar to $\triangle CDP$. Hence:

$$\frac{AP}{DP} = \frac{BP}{CP}$$

$$\therefore \quad AP \times CP = BP \times DP$$

It is left to the student to try to word the converses of these results. They will be found to be useful.

I. Percentage changes

Regular polls and surveys have shown that percentages are one of the least understood arithmetical topics that have to be used in commercial activities by the general population. In the case of Value Added Tax [1], a percentage (currently set at 20%) is added by the store to the price they want to collect. Suppose they wish to receive £12.99. They add 20% to this, namely £2.60, charge the customer £15.59 and pass over £2.60 to the government. Note that we can also understand this calculation as multiplying the original price that the shop wishes to collect by $\frac{120}{100}$ to determine the customer price.

A useful equation for understanding this is as follows:

$$\text{Original price} \times \frac{120}{100} = \text{Charged price}$$

In practice the trader will probably charge a price which seems attractive to the customer. Suppose the trader advertises an article at £19.99. We now have:

$$\text{Charged price} \div \frac{120}{100} = \text{Original price}$$

The trader collects £16.66 for the store and passes on £19.99 − £16.66 = £3.33 tax. Note that you must not multiply by $\frac{80}{100}$ to find how much the trader collects. To do so, the trader would be cheating themself and paying too much tax to the government.

> There is also a term "percentage point" in use. This describes arithmetic differences between percentages. For example, if we think of an item as costing 120 percentage points in a shop, the shop-keeper gets 100 percentage points while the government gets 20 percentage points. As another example, if the percentage of people accessing NHS emergency services within 4 hours of arrival was 60% in 2002 but 75% in 2012, there has been an improvement of 15 percentage points but the percentage rise was 25% over the ten year period. This is even more marked in a case where a percentage of the population rises from a small value, say 4% to 5%. The percentage rise is 25% but the there has only been a change of 1 percentage point.
>
> This distinction causes much confusion, especially in the media. It is wise to be suspicious of percentage statistics when applied to small numbers of cases. Always question the base on which statistics are applied. Many presenters try to bamboozle their audience by quoting percentages which are, at best, irrelevant to the argument they are trying to make and, at worst, designed to make a problem seem worse than it actually is.

[1] A VAT registered trader claims back from the government the VAT on everything bought, but then pays the government the VAT on everything sold. In this way, the trader only pays the extra VAT on the markup, i.e. the increase in the value of the product. This is the reason for the name Value Added Tax.

J. Rules of inequalities

The symbols $<, \leq, >$ and $\geq$ are often mistrusted. Many students avoid them in their solutions, substituting = before trying to convince themselves (and the marker or other readers) later on that they have got the direction ($<$ or $>$) the right way round.

As a general rule, substitute in simple numbers as you go in order to check you are on track.

The rules are as follows:

$a < b$ is the same as $b > a$

For example, $3 < 5$ is the same as $5 > 3$.

If $a < b$, then $a + x < b + x$ for any number x.

For example, since $3 < 5$ if follows that $-4 < -2$ by adding -7 to both sides of the inequality.
It works for subtraction. If $a < b$, then $a - x < b - x$ for any number x.

If $a < b$, then $ax < bx$ for any *positive* number x, but $ax > bx$ if x is *negative*.

Since $3 < 5$, it follows that $12 < 20$ by multiplying by 4.
But if we multiply by -2, say, we obtain $-6 > -10$.

If $a < b$, then $-a > -b$.

Changing signs is the equivalent of multiplying by -1.

If $a < b$, then $\frac{1}{a} > \frac{1}{b}$ provided a and b are *both* positive or both negative.

For example $3 < 5$ yields $\frac{1}{3} > \frac{1}{5}$. Moreover, but not so obvious, $-5 < -3$ yields $-\frac{1}{5} > -\frac{1}{3}$.

In summary you may add to or subtract from both sides of an inequality just as with an equation. You may also preserve the inequality by multiplying or dividing both sides by a positive number. Until you build confidence in working inequalities, you are advised to manipulate the algebra so that these are the only operations you have to apply.

There are some situations which are likely to catch you out. The most frequent are cases which involve multiplying or dividing by numbers which may or may not be negative. Here are some examples.

(a) Let $x < 3$. Is it true that $x^2 < 9$? Perhaps. If $x = 1$, then $x^2 < 9$ is true. But if $x = -5$, then it is not true since $x^2 = 25$. So we can *not* write:

$$\begin{aligned} & x < 3 \\ \therefore \quad & x^2 < 9 \qquad \text{THIS IS WRONG} \end{aligned}$$

(b) Conversely if $x^2 > 9$, we can easily overlook solutions like $x < -3$. This is very similar to saying that if $x^2 = 9$, then $x = 3$. We ignore the case $x = -3$ at our peril! We should write:

$$\begin{aligned} & x^2 > 9 \\ \therefore \quad & x < -3 \text{ or } x > 3 \end{aligned}$$

Final rule. Always watch out for hidden negative multipliers.

K. Index, subscript or suffix notation

If we have a sequence, we could name the terms of the sequence with letters in alphabetical order as $a, b, c, \ldots$, but we would soon run out of letters. Instead, it is useful to use one letter for the whole sequence and to a attach a small number next to it, written just below the line. This is called a *subscript* or *suffix*. It should not be confused with *index* such as the 5 in 2^5 which represents the 5th power of 2, producing the value 32.

Here is an example using the Fibonacci sequence, which is defined by adding two consecutive terms to yield the next term. We start with 1 and 1.

$$F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ F_5 = 5,\ F_6 = 8,\ F_7 = 13,\ F_8 = 21,\ \ldots$$

Sometimes it is useful to include the term F_0.
In this case, we take its value as 0 so that $F_0 + F_1 = F_2$.

Some examples of the use of this notation:
(a) F_8 is 5 positions along from F_3.
(b) F_8^2 means the square of F_8 and has the value $21^2 = 441$.
(c) F_n means a term in the general position n. That means that F_{n-1} is the term before F_n, F_{n+1} is the term after F_n, F_{n+2} is the term after that and so on.
(d) F_{2n} means the term in the $2n$-th position, which is the term in the nth even position. Similarly F_{2n-1} is the term in the nth odd position.
(e) We can see that F_5^2 is one more than $F_4 \times F_6$. We normally omit $\times$ and write

$$F_5^2 = F_4F_6 + 1$$

Check that it is true that F_n^2 is one more or one less that $F_{n-1}F_{n+1}$ for some other values of n.
It turns out:

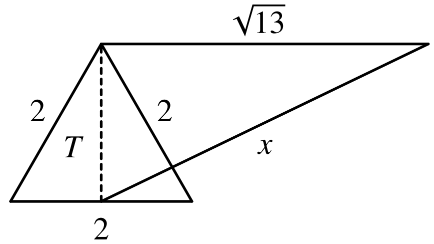
$$F_n^2 = F_{n-1}F_{n+1} + (-1)^{n+1}$$

You might like to prove this.

L. Surds

Numbers like $\sqrt{5}$ are called *surds*. The best way to treat them is like letters in algebra. It can help to replace a surd with a letter as it may make the algebra easier to read.

Suppose you put c in place of $\sqrt{5}$. What happens when you find c^2? It comes to exactly 5 because $(\sqrt{5})^2 = 5$. What about c^3? This is $c^2 \times c = 5c$ or $5\sqrt{5}$. Note that $c^4 = 25$ exactly.



Consider an equilateral triangle with its height drawn in as shown by the dotted line. This line forms one side of the right-angled triangle labelled T. The lengths of the sides are shown in the diagram. It is left to the reader to use Pythagoras's Theorem to check that the height of the equilateral triangle is $\sqrt{3}$. Applying the theorem again to the large triangle, we find that its hypotenuse has length $x = \sqrt{16}$ which is exactly 4. Changing surds into decimals can never *prove* this result.

M. Modular arithmetic

In combination with properties of prime numbers, this topic is crucial to digital encryption.

(a) Modular arithmetic means that you are doing arithmetic on a finite set of integers $\{0, 1, 2, \ldots, n-1\}$. An example is to use {0, 1, 2, 3, 4, 5, 6} to represent the days of the week. Let's say that 3 represents Wednesday. Thus 18 days on from Wednesday gives 21. The number 21 can be changed back to a number in the set by dividing by 7 and using the remainder which is 0. Hence 18 days on from Wednesday is a Sunday.

(b) Useful ideas related to the days of the week in this arithmetic are: May moves on 3 days in the week because 31 is equivalent to 3; a year moves on 1 day because 365 is equivalent to 1 and a leap-year moves on 2 days because 366 is equivalent to 2.
Hence May 1st 2013 was on a Wednesday so June 1st 2013 was on a Saturday. Also January 1st 2012 was on a Sunday so January 1st 2013 was on a Tuesday.

(c) One way to think of this is that all the numbers $\{\ldots, -14, -7, 0, 7, 14, 21, 28, \ldots\}$ are represented by 0. Similarly $\{\ldots, -13, -6, 1, 8, 15, 22, 29, \ldots\}$ are represented by 1 and so on. When we write

$$3 + 6 \equiv 2 \pmod 7$$

we mean that adding a number in the 3 set to a number in the 6 set must give an answer in the 2 set. Negative numbers are included but, for example, -12 is in the 2 set and not in the 5 set with $+12$. An algebraic expression for numbers in the 2 set is $7k + 2$ where k runs through all the integers. To check that, for example, 43178 lies in the 2 set, divide 43178 by 7 and find the remainder which is 2.

(d) We often make little attempt to show the difference between a number such as 2 in the 2 set. Different authors use slightly different notation to that introduced in the previous paragraph: $3 + 6 \equiv 2 \pmod 7$, $3 + 6 = 2 \pmod 7$, $3 + 6 \equiv 2$ if we are working modulo 7 (*) throughout the calculation or even $3 + 6 \equiv 2$. The first method is probably the clearest, retaining the (mod ..) notation unless there is no confusion about what the modulus of the arithmetic is.

[* The word modulo is used as short for 'to the modulus of': derived from Latin grammar. Modular is the associated adjective.]

(e) How do we deal with negative answers? Suppose we wanted $-43178 \pmod 7$. Divide by 7 and obtain the remainder -2. Now add 7. This shifts the number into the $\{0, \ldots, 6\}$ set so we use the number 5.

(f) Addition, subtraction and multiplication can be peformed as normal with modular arithmetic, simplifying the results as above. A subtraction table modulo 5 and a multiplication table modulo 6 show the principles. In the subtraction table the number at the head of each column is taken as x, the number at the left of each row as y and the result is $x - y$.

mod 5	0	1	2	3	4
0	0	1	2	3	4
1	4	0	1	2	3
2	3	4	0	1	2
3	2	3	4	0	1
4	1	2	3	4	0

mod 6	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

We note $2-4 \equiv 3 \pmod 5$ in the first table because we do not want to write the negative answer -2.

Division is not always possible ...

Division is not always possible with modular arithmetic. Instead of writing something like $1 \div 3 \pmod{6}$, we try solving the equation $3x \equiv 1 \pmod{6}$. Using the table above, we find that this equation has no solution. On the other hand, the equation $3x \equiv 3 \pmod{6}$ has three solutions: $x \equiv 1, x \equiv 3$ and $x \equiv 5$. Modular arithmetic can be surprising!

(g) Examining the properties of the units places of numbers (in decimal) can be likened to working in arithmetic modulo 10. This is a nice way to look at the units places of squares, cubes and higher powers of integers. Being able to reduce large numbers to manageable size can make calculations in some problems remarkably easy.

(h) Some interesting results emerge from this arithmetic. This is Fermat's little theorem:

$$a^p \equiv a \pmod{p}$$

assuming p is a prime number. The mathematician Euler devised a generalisation of this to cover all moduli.

N. The Greek alphabet

This is often used in mathematics when the writer runs out of (Latin) letters or wishes to use a difference style of letter.

There are twenty-four letters in the Greek alphabet, usually written in this order:

Α	α	**al**pha	a	Ι	ι	**i**ota	i	Ρ	ρ, ϱ	**r**ho	r
Β	β	**b**eta	b	Κ	$\kappa, \varkappa$	**k**appa	k	Σ	σ, ς	**s**igma	s
Γ	γ	**g**amma	g	Λ	λ	**l**ambda	l	Τ	τ	**t**au	t
Δ	δ	**d**elta	d	Μ	μ	**m**u	m	Υ	υ	**u**psilon	u
Ε	ϵ, ε	**e**psilon	e	Ν	ν	**n**u	n	Φ	ϕ, φ	**ph**i	f
Ζ	ζ	**z**eta	z	Ξ	ξ	**x**i	x	Χ	χ	**ch**i	c
Η	η	**e**ta	h	Ο	o	**o**micron	o	Ψ	ψ	**ps**i	y
Θ	θ, ϑ	**th**eta	q	Π	π, ϖ	**p**i	p	Ω	ω	**o**mega	w

In this table, the first column is the capital letter and the second column shows the lower case letters. (Seven letters are shown with common variant forms, and they are as follows: the two variants of the letters epsilon, theta and phi are both commonly found in mathematics books and papers; the second variant of the letters kappa, pi and rho are found, but are significantly rarer than the first form; finally, the variant form of sigma is very rare in mathematics books, and is used in Greek when a sigma appears at the end of a word.)

The third column is the name of the letter, with the bold-faced letter being the English pseudo-equivalent. While many pronounce the 'e' in 'eta' as in 'least', classical scholars often use an 'air' sound to pronounce the 'e'. The vowels in beta, zeta and theta are pronounced as in eta, and the 'o' in omega is as in 'bold'.

The fourth column shows the keys which correspond to these letters in the standard keyboard mapping for the Symbol font on Windows.