United Kingdom
Mathematics Trust

Mentoring Scheme

Supported by Oxford Asset Management

Hanna Neumann

Sheet 1

Questions

This programme of the Mentoring Scheme is named after Hanna Neumann (1914–1971).

See http://www-history.mcs.st-andrews.ac.uk/Biographies/Neumann_Hanna.html for more information, or search "Hanna Neumann mathematician" to find a wider range of sources.

Enquiries about the Mentoring Scheme should be sent to:

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1. Set the scene for your mentoring work this year by discovering some facts about the life and work of Hanna Neumann.

2. A pair of twin primes consists of two successive odd integers p and $p + 2$ that are both primes. The integer between the two primes must, of course, be even. How many such integers are perfect squares? Justify your answer.

3. How many positive integers less than or equal to 1000 are not divisible by any of 2, 3 or 5?

4. Will, Xavier, Yolanda and Zara are sixth-formers who like to work on maths problems together. Sometimes they tease their teacher by handing in a single, typed set of solutions. When questioned on one occasion, they make the following set of statements, each of which is either true or false.

Will:	It was Xavier.
Xavier:	It was Zara.
Yolanda:	It was not me.
Zara:	What Xavier says is wrong.

a) Let us suppose that exactly one of the four is lying and the other three are telling the truth. Who is lying? Who typed out the solutions?

b) Let us now suppose that exactly one of the four is telling the truth and the other three are lying. Who is telling the truth? Who typed out the solutions?

c) Now suppose that two of the statements are true and two are false. Who are the people who could have typed out the solutions?

d) Without prior knowledge of the number of true statements, what can we say about the identity of the typist?

5. Prove that for all positive real numbers a, b, c:

a) $(a + b)^2 \geqslant 4ab$;

b) $\frac{a+b+c}{2} \geqslant \frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b}$.

6. An odd number of soldiers are stationed in a field, in such a way that all their pairwise distances from each other are distinct. (Note this means that, if you make a list of all the distances between each possible pair of soldiers, then all the distances in the list are distinct.) Each soldier is told to keep an eye on the soldier that is nearest to him. Prove that at least one soldier is not being watched.

7. Two touching circles S and T share a common tangent which meets S at A and T at B. Let AP be a diameter of S and let the tangent from P to T touch it at Q. Show that $AP = PQ$.

8. *Something for you to investigate*

Twenty-first century primes: find all the primes between 2001 and 2100 inclusive, justifying your choice in each case.

Are any of your primes Mersenne numbers, of the form $2^n - 1$?

Are any of your primes Fermat numbers, of the form $2^{2^n} + 1$?

Have you found any twin primes, consecutive odd numbers which are both prime?

