

# JUNIOR MATHEMATICAL CHALLENGE

Organised by the United Kingdom Mathematics Trust



## SOLUTIONS AND INVESTIGATIONS

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some exercises for further investigation. We welcome comments on these solutions. Please send them to [enquiry@ukmt.org.uk](mailto:enquiry@ukmt.org.uk).

The Junior Mathematical Challenge (JMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the JMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. So usually for each question we have included a complete solution which does not use the fact that one of the given alternatives is correct. We aim to give full solutions with all steps explained. We hope that these solutions can be used as a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem, for example, in the Junior Mathematical Olympiad and similar competitions.

These solutions may be used freely within your school or college. You may, without further permission, post these solutions on a website that is accessible only to staff and students of the school or college, print out and distribute copies within the school or college, and use them in the classroom. If you wish to use them in any other way, please consult us. © UKMT April 2017

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1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25  
E D D E C D A B D E B D C B A D B A A B C B E E E

1. Which of the following calculations gives the largest answer?

- A  $2 - 1$       B  $2 \div 1$       C  $2 \times 1$       D  $1 \times 2$       E  $2 + 1$

**SOLUTION**      **E**

We have

$$2 - 1 = 1,$$

$$2 \div 1 = 2,$$

$$2 \times 1 = 2,$$

$$1 \times 2 = 2,$$

and

$$2 + 1 = 3.$$

We therefore see that option E gives the largest value.

2. Nadiya is baking a cake. The recipe says that her cake should be baked in the oven for 1 hour and 35 minutes. She puts the cake in the oven at 11:40 am.

At what time should she take the cake out of the oven?

- A 12:15 pm      B 12:40 pm      C 1:05 pm      D 1:15 pm      E 2:15 pm

**SOLUTION**      **D**

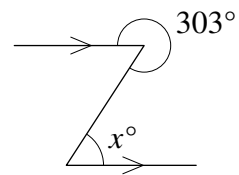
1 hour and 35 minutes is 1 hour plus 20 minutes plus 15 minutes.

One hour after 11:40 am the time is 12:40 pm. After a further 20 minutes, the time is 1 pm. Then 15 minutes later it is 1:15 pm.

Therefore 1 hour and 35 minutes after 11:40 am the time is 1:15 pm.

3. What is the value of  $x$  ?

- A 43      B 47      C 53      D 57      E 67



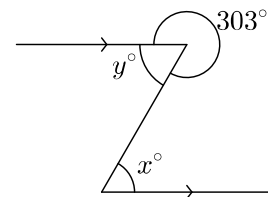
**SOLUTION**      **D**

Let the angle shown be  $y^\circ$ .

Because the angles at a point total  $360^\circ$ , we have  $y + 303 = 360$ .  
Therefore  $y = 360 - 303 = 57$ .

Because the alternate angles formed by a line which cuts a pair of parallel lines are equal,  $x = y$ .

Hence  $x = 57$ .



4. A download is 95% complete.

What fraction is yet to be downloaded?

A  $\frac{1}{2}$

B  $\frac{1}{5}$

C  $\frac{1}{9}$

D  $\frac{1}{10}$

E  $\frac{1}{20}$

**SOLUTION**

**E**

When the download is 95% complete, 5% remains to be downloaded.

Now 5% means 5 parts in 100. Therefore, expressed as a fraction, 5% is

$$\frac{5}{100} = \frac{1}{20}.$$

5. What is the value of  $201 \times 7 - 7 \times 102$  ?

A 142 800

B 793

C 693

D 607

E 0

**SOLUTION**

**C**

**METHOD 1**

We have  $201 \times 7 - 7 \times 102 = 1407 - 714 = 693$ .

**NOTE**

This straightforward method does not involve too much hard work. However, as both the products  $201 \times 7$  and  $7 \times 102$  involve the factor 7, there is an alternative method in which we first take out this common factor.

**METHOD 2**

We have

$$\begin{aligned} 201 \times 7 - 7 \times 102 &= 7 \times (201 - 102) \\ &= 7 \times 99 \\ &= 7 \times (100 - 1) \\ &= 7 \times 100 - 7 \times 1 \\ &= 700 - 7 \\ &= 693. \end{aligned}$$

**FOR INVESTIGATION**

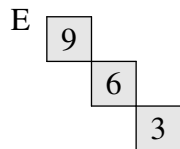
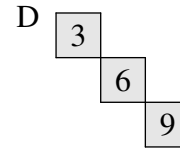
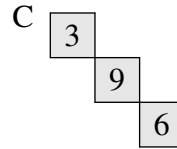
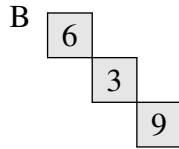
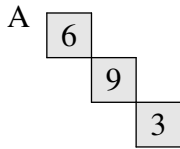
**5.1** Evaluate (without using a calculator)

(a)  $283 \times 17 - 17 \times 184$ ,      (b)  $2017 \times 819 - 819 \times 1918$ .

6. In a magic square, the numbers in each row, each column and the two main diagonals have the same total. This magic square uses the integers 2 to 10.

	10	5
8		4
7	2	

Which of the following are the missing cells?



SOLUTION

**D**

METHOD 1

The row totals given by the numbers shown in the square in the question are  $10 + 5 = 15$ ,  $8 + 4 = 12$  and  $7 + 2 = 9$ . Therefore to make all the row totals the same, the number we add to the second row must be three more than the number we add to the first row, and the number we add to the third row must be three more than the number we add to the second row.

Therefore the only possibility for the missing cells among the given options is that given by option D.

3	10	5
8	6	4
7	2	9

As the square on the right shows, when we add these missing cells, the row totals, the column totals and the totals on the two main diagonals are all equal to 18. So option D is correct.

METHOD 2

The numbers in the completed magic square have total  $2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 54$ . This is the sum of the three row totals. Since these row totals are equal, each row total in the magic square is  $54 \div 3 = 18$ . To achieve these totals from the numbers already shown in the square in the question, the numbers that need to be added to the three rows are 3, 6 and 9 in this order from top to bottom. So the correct option is D.

FOR INVESTIGATION

6.1 A  $4 \times 4$  magic square. Arrange the integers from 1 to 16 inclusive in the  $4 \times 4$  grid shown, with one integer in each small square, so that the numbers in each row, each column and the two main diagonals have the same total.


6.2 To answer 6.1, it helps to know what the common row, column and main diagonal totals are. Can you work out what this will be?

7. If you work out the values of the following expressions and then place them in increasing numerical order, which comes in the middle?

A  $\frac{2}{3} + \frac{4}{5}$

B  $\frac{2}{3} \times \frac{4}{5}$

C  $\frac{3}{2} + \frac{5}{4}$

D  $\frac{2}{3} \div \frac{4}{5}$

E  $\frac{3}{2} \times \frac{5}{4}$

**SOLUTION**

**A**

We evaluate each expression in turn, and we write the answers as fractions with the same denominator so as to make it easy to arrange them in increasing order.

We have

$$\frac{2}{3} + \frac{4}{5} = \frac{10 + 12}{15} = \frac{22}{15} = \frac{176}{120}$$

$$\frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5} = \frac{8}{15} = \frac{64}{120}$$

$$\frac{3}{2} + \frac{5}{4} = \frac{12 + 10}{8} = \frac{22}{8} = \frac{330}{120}$$

$$\frac{2}{3} \div \frac{4}{5} = \frac{2}{3} \times \frac{5}{4} = \frac{10}{12} = \frac{100}{120}$$

$$\frac{3}{2} \times \frac{5}{4} = \frac{3 \times 5}{2 \times 4} = \frac{15}{8} = \frac{225}{120}$$

We now see that when the fractions are arranged in increasing numerical order the order is

$$\frac{2}{3} \times \frac{4}{5}, \quad \frac{2}{3} \div \frac{4}{5}, \quad \frac{2}{3} + \frac{4}{5}, \quad \frac{3}{2} \times \frac{5}{4}, \quad \frac{3}{2} + \frac{5}{4}.$$

Therefore it is the expression

$$\frac{2}{3} + \frac{4}{5}$$

whose value is in the middle.

**FOR INVESTIGATION**

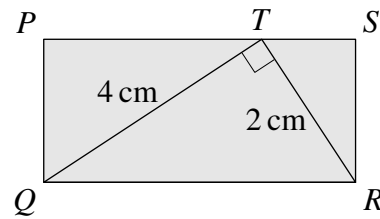
7.1 Arrange the following fractions in increasing order.

$$\frac{3}{5}, \quad \frac{5}{9}, \quad \frac{7}{12}, \quad \frac{11}{18}, \quad \frac{13}{24}, \quad \frac{19}{30}.$$

7.2 (a) Find a fraction  $\frac{a}{b}$ , where  $a$  and  $b$  are positive integers with no common factor other than 1, so that when the fractions  $\frac{2}{3}$ ,  $\frac{3}{4}$  and  $\frac{a}{b}$  are arranged in increasing order, the fraction  $\frac{a}{b}$  comes in the middle.

(b) Find the fraction  $\frac{a}{b}$  which satisfies the requirement of (a), and where  $a + b$  is as small as possible. What do you notice?

8. The diagram shows a rectangle  $PQRS$  and  $T$  is a point on  $PS$  such that  $QT$  is perpendicular to  $RT$ . The length of  $QT$  is 4 cm. The length of  $RT$  is 2 cm.



What is the area of the rectangle  $PQRS$ ?

- A  $6 \text{ cm}^2$       B  $8 \text{ cm}^2$       C  $10 \text{ cm}^2$   
 D  $12 \text{ cm}^2$       E  $16 \text{ cm}^2$

**SOLUTION**

**B**

The area of a triangle is given by the formula

$$\frac{1}{2}(\text{base} \times \text{height})$$

whereas the area of a rectangle is given by

$$\text{base} \times \text{height}.$$

The triangle  $QRT$  and the rectangle  $PQRS$  have the same base,  $QR$ , and the same height.

It follows that the area of the rectangle  $PQRS$  is twice the area of the triangle  $QRT$ .

Because  $QT$  is perpendicular to  $RT$ , if we take  $QT$  as the base of the triangle  $QRT$ , its height is given by the length of  $RT$ . Hence the area of the triangle  $QRT$  is

$$\frac{1}{2}(QT \times RT) = \frac{1}{2}(4 \text{ cm} \times 2 \text{ cm}) = 4 \text{ cm}^2.$$

Hence the area of the rectangle  $PQRS$  is

$$2 \times 4 \text{ cm}^2 = 8 \text{ cm}^2.$$

9. In William Shakespeare's play *As You Like It*, Rosalind speaks to Orlando about, "He that will divide a minute into a thousand parts".

Which of the following is equal to the number of seconds in one thousandth of one minute?

- A 0.24      B 0.6      C 0.024      D 0.06      E 0.006

**SOLUTION**

**D**

There are 60 seconds in one minute. Therefore the number of seconds in one thousandth of a minute is

$$\frac{60}{1000} = \frac{6}{100} = 0.06.$$

**FOR INVESTIGATION**

- 9.1 Are there more seconds in one millionth of one day than in one thousandth of one minute?

**10.** Which of the following integers is not a multiple of 45?

A 765

B 675

C 585

D 495

E 305

**SOLUTION**

**E**

Because  $45 = 5 \times 9$ , if an integer is divisible by 45, it is divisible both by 5 and by 9. Conversely, if an integer is divisible both by 5 and by 9, then, because 5 and 9 have no common factors other than 1, the integer is also divisible by 45.

Each of the integers 765, 675, 585, 495 and 305 has units digit 5, and therefore is divisible by 5.

The test for whether an integer is divisible by 9 is whether the sum of its digits is divisible by 9.

We see that  $7 + 6 + 5 = 18$ ,  $6 + 7 + 5 = 18$ ,  $5 + 8 + 5 = 18$  and  $4 + 9 + 5 = 18$ . Because 18 is divisible by 9 we deduce that all of the first four integers given as options are divisible by 9. However  $3 + 0 + 5 = 8$  and so the sum of the digits of 305 is not divisible by 9.

We deduce that 305 is the only one of the given options that is not divisible by 45.

**FOR INVESTIGATION**

**10.1** Explain why a number whose units digit is 5 is divisible by 5.

**10.2** Is it true that every integer which is divisible by 5 has 5 as its units digit?

**10.3** In the above solution we said that the test for whether an integer is divisible by 9 is whether the sum of its digits is divisible by 9. Explain why this test always gives the correct answer. [Hint: show that when you divide an integer by 9, and you also divide the sum of its digits by 9, you will obtain the same answer.]

**10.4** Show that, similarly, the test for whether an integer is divisible by 3 is whether the sum of its digits is divisible by 3.

**10.5** What are the remainders when the following integers are divided by 9?

(a) 111 111

(b) 111 111 111

(c) 1 111 111 111 111 111 111

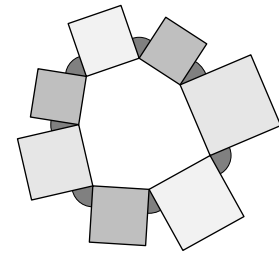
**10.6** Consider the integer which when written out in full is a string of 2017 1s. What is the remainder when this number is divided by 9?

**10.7** Can you find a test which uses the digits of an integer to decide whether it is divisible by 11?

11. Seven squares are drawn on the sides of a heptagon so that they are outside the heptagon, as shown in the diagram.

What is the sum of the seven marked angles?

- A  $315^\circ$       B  $360^\circ$       C  $420^\circ$       D  $450^\circ$   
 E  $630^\circ$



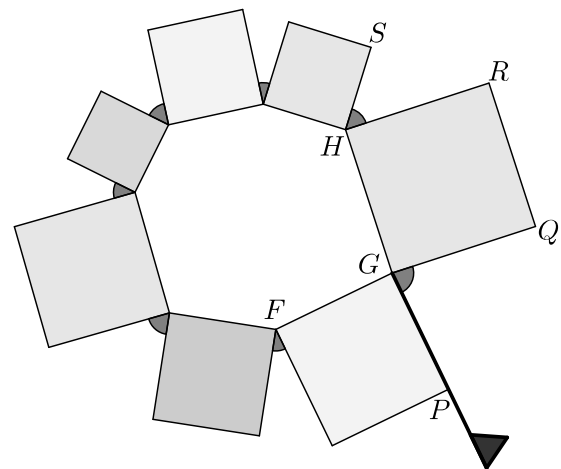
SOLUTION

**B**

METHOD 1

In the figure we have labelled some of the vertices so that we may refer to them.

Suppose that there is a flag whose pole is in the direction of  $GP$  and pointing as shown. Consider the effect of carrying out the following operations. First rotate the flag anti-clockwise about  $G$  through  $\angle PGQ$  so that now its pole lies along  $GQ$ . Next slide the flag without rotation so that its pole lies along  $HR$ . Next rotate the pole about  $H$  through  $\angle RHS$  so that it lies along  $HS$ , and so on, until the flag returns to its original position.



The total angle that the flag has turned through is the sum of the seven marked angles. But in returning to its original position the flag has completed a full rotation of  $360^\circ$ .

Therefore the sum of the seven marked angles is  $360^\circ$ .

METHOD 2

We use the same labelling of the vertices that we used in Method 1. We let the sum of the marked angles be  $X^\circ$ .

The sum of the angles at the vertex  $G$  is  $360^\circ$ . Now  $\angle FGP$  and  $\angle QGH$  are angles of a square, and so are each  $90^\circ$ . It follows that  $\angle PGQ + \angle HGF = 360^\circ - 90^\circ - 90^\circ = 180^\circ$ . Therefore

$$\angle PGQ = 180^\circ - \angle HGF.$$

A similar equation holds for each of the seven marked angles. It follows, by adding the seven equations obtained in this way, that

$$X^\circ = 7 \times 180^\circ - \text{the sum of the interior angles of the heptagon.}$$

The sum of the interior angles of a polygon with  $n$  vertices is  $(n - 2) \times 180^\circ$ . Therefore the sum of the interior angles of a heptagon is  $5 \times 180^\circ$ . Therefore

$$X^\circ = 7 \times 180^\circ - 5 \times 180^\circ = 2 \times 180^\circ = 360^\circ.$$



**12.** Last year, at the school where Gill teaches Mathematics, 315 out of the 600 pupils were girls. This year, the number of pupils in the school has increased to 640. The proportion of girls is the same as it was last year.

How many girls are there at the school this year?

- A 339                      B 338                      C 337                      D 336                      E 335

**SOLUTION**

**D**

The fraction of girls in the school last year was  $\frac{315}{600}$ . By dividing both the numerator and denominator by 3 and then by 5 we see that

$$\frac{315}{600} = \frac{105}{200} = \frac{21}{40}.$$

The proportion of girls this year is the same. Therefore, of the 640 pupils in the school this year, the number that are girls is

$$\frac{21}{40} \times 640 = 21 \times 16 = 336.$$

**13.** Consider the following three statements.

- (i) Doubling a positive number always makes it larger.
- (ii) Squaring a positive number always makes it larger.
- (iii) Taking the positive square root of a number always makes it smaller.

Which statements are true?

- A All three                      B None                      C Only (i)                      D (i) and (ii)  
E (ii) and (iii)

**SOLUTION**

**C**

Statement (i) is true, because, if  $x$  is a positive number, then

$$2x = x + x > x.$$

Statement (ii) is false. If a positive number which is less than 1 is squared, then the answer is smaller than the original number. For example, with  $x = \frac{1}{2}$ , we have

$$x^2 = \frac{1}{4} < \frac{1}{2}.$$

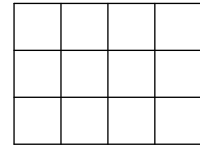
Similarly, if we take the square root of a positive number that is less than 1, then the answer is larger than the original number. For example

$$\sqrt{\frac{1}{4}} = \frac{1}{2} > \frac{1}{4}.$$

Hence statement (iii) is false.

Therefore statement (i) is the only one that is true.

**14.** Mathias is given a grid of twelve small squares. He is asked to shade grey exactly four of the small squares so that his grid has two lines of reflection symmetry.



How many different grids could he produce?

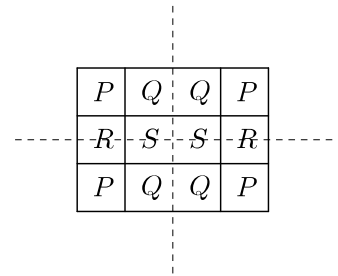
- A 2      B 3      C 4      D 5      E 6

**SOLUTION**

**B**

The two lines of reflection symmetry of the grid are shown in the figure on the right as broken lines.

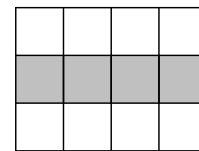
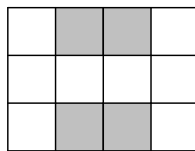
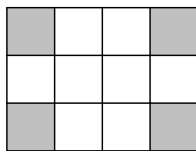
We see from this that if the grid with some of the small squares shaded grey has both these lines of reflection symmetry, then all the four squares labelled *P* must be the same colour. Similarly, all the four squares labelled *Q* must be the same colour, the two squares labelled *R* must be the same colour, and the two squares labelled *S* must be the same colour.



It follows that there are only three ways in which Mathias can shade exactly four of the small squares of the grid grey so that the result has two lines of reflection symmetry. These are

- (i) shade grey all the squares labelled *P*, and no others,
- (ii) shade grey all the squares labelled *Q*, and no others,
- and (iii) shade grey all the squares labelled *R* and all those labelled *S*, and no others.

The three different grids that Mathias could produce are shown in the figure below.



**FOR INVESTIGATION**

- 14.1** Suppose the restriction that Mathias has to shade exactly four of the small squares grey is dropped. In how many ways can he shade some of these squares grey, leaving any others white, so that his grid has two lines of reflection symmetry? (Include the case where no squares are coloured grey.)
- 14.2** In how many ways can Mathias colour some of the small squares of the grid grey, leaving any others white, so that the grid has a rotational symmetry of order 2? (Include the case where no squares are coloured grey.)
- 14.3** In how many ways can Mathias colour some of the squares red and some blue, leaving any others white, so that his grid has two lines of reflection symmetry? (Include the cases where only one colour is used and where every square is left white.)
- 14.4** Suppose Mathias has paint of *k* different colours. He colours each of these squares using one of these colours, but he doesn't have to use all *k* colours. In how many ways can he do this so that his grid has two lines of reflection symmetry?

- 15.** What is the remainder when the square of 49 is divided by the square root of 49?  
 A 0                      B 2                      C 3                      D 4                      E 7

**SOLUTION**     **A**

Because  $49 = 7 \times 7$ , it follows both that  $\sqrt{49} = 7$  and that 7 is a factor of 49.

Therefore 7 is also a factor of  $49^2$ .

Hence the remainder when the square of 49 is divided by the square root of 49 is 0.

**FOR INVESTIGATION**

- 15.1** Find, without using a calculator, the value of  $49^2 \div \sqrt{49}$ .
- 15.2** Explain why it is true that for each positive integer  $n$ , the remainder when  $(n^2)^2$  is divided by  $\sqrt{n^2}$  is 0.
- 15.3** Give a general formula for  $(n^2)^2 \div \sqrt{n^2}$ , in the case where  $n$  is a positive integer.
- 15.4** Is the formula that you gave in answer to 15.3 also correct in the case where  $n$  is a negative integer?

- 16.** In New Threeland there are three coins: the 2p; the 5p; and one other.  
 The smallest number of coins needed to make 13p is three.  
 The smallest number of coins needed to make 19p is three.  
 What is the value of the third type of coin?  
 A 4p                      B 6p                      C 7p                      D 9p                      E 12p

**SOLUTION**     **D**

We have

$$2p + 2p + 9p = 13p$$

and  $5p + 5p + 9p = 19p.$

Therefore if the third type of coin is worth 9p, both 13p and 19p may be made using three coins. Also, in this case we see that we cannot make either 13p or 19p with just two of the coins. So when the third type of coin is worth 9p, the smallest number of coins needed to make 13p and 19p is three.

In the context of JMC, where we can assume that just one of the options is correct, we can now deduce that option D is the correct one.

**FOR INVESTIGATION**

- 16.1** A full solution requires a proof that 9p is the only possible value for the third type of coin. Give an argument to show this.

**17.** I add up all even numbers between 1 and 101. Then from my total I subtract all odd numbers between 0 and 100.

What is the result?

- A 0                      B 50                      C 100                      D 255                      E 2525

**SOLUTION**    **B**

The sum we are asked to work out is

$$(2 + 4 + 6 + \dots + 96 + 98 + 100) - (1 + 3 + 5 + \dots + 95 + 97 + 99),$$

where ... indicates that we have left out some numbers because of lack of space. We may rewrite this expression as

$$(2 - 1) + (4 - 3) + (6 - 5) + \dots + (96 - 95) + (98 - 97) + (100 - 99),$$

which is equivalent to

$$1 + 1 + 1 + \dots + 1 + 1 + 1.$$

There are fifty 1s in this expression, and hence the sum is 50.

**18.** What is the sum of the digits in the completed crossnumber?

<p><b>ACROSS</b></p> <p>1. A cube</p> <p>3. A power of 11</p>	<p><b>DOWN</b></p> <p>2. A square</p>
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- A 25                      B 29                      C 32                      D 34                      E 35

**SOLUTION**    **A**

The answer to 3 Across is a 5-digit power of 11. We have  $11^2 = 121$ ,  $11^3 = 1331$ ,  $11^4 = 14641$ . Any higher power of 11 has more than 5 digits. We deduce that 3 Across is 14641.

It follows that 2 Down is a 2-digit square with units digit 4. Hence 2 Down is 64.

We now see that 1 Across is a 3-digit cube with units digit 6. The only such 3-digit cube is 216 ( $= 6^3$ ).

Hence the completed crossnumber is as shown in the figure.

We see that the sum of the digits in the completed crossnumber is  $2 + 1 + 6 + 1 + 4 + 6 + 4 + 1 = 25$ .



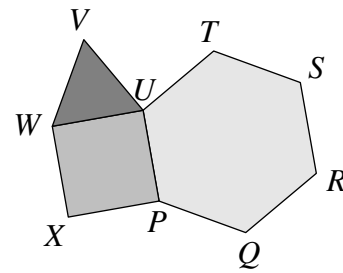
**FOR INVESTIGATION**

- 18.1** Find the value of  $11^5$  and check that it has more than 5 digits.
- 18.2** List the 2-digit squares and check that 64 is the only one which has 4 as its units digit.
- 18.3** List the 3-digit cubes and check that 216 is the only one which has 6 as its units digit.

19. The diagram shows a regular hexagon  $PQRSTU$ , a square  $PUWX$  and an equilateral triangle  $UVW$ .

What is the angle  $TVU$  ?

- A  $45^\circ$       B  $42^\circ$       C  $39^\circ$       D  $36^\circ$   
 E  $33^\circ$



**SOLUTION**      **A**

The total of the angles at the point  $U$  is  $360^\circ$ . Now  $\angle WUP = 90^\circ$ , as it is the angle of a square;  $\angle VUW = 60^\circ$ , as it is the angle of an equilateral triangle; and  $\angle PUT = 120^\circ$ , as it is the angle of a regular hexagon. It follows that  $\angle TUV = 360^\circ - 90^\circ - 60^\circ - 120^\circ = 90^\circ$ .

Because  $PQRSTU$  is a regular hexagon

$$TU = PU.$$

Because  $PUWX$  is a square

$$PU = UW.$$

Because  $UVW$  is an equilateral triangle

$$WU = UV.$$

It follows that  $TU = UV$ .

We have therefore shown that  $VUT$  is a right-angled isosceles triangle. Therefore the angles other than the right angle in this triangle are each  $45^\circ$ . Hence  $\angle TVU = 45^\circ$ .

**FOR INVESTIGATION**

19.1 Explain why it follows from the fact that  $VUT$  is an isosceles triangle in which  $\angle TUV = 90^\circ$ , that  $\angle TVU = 45^\circ$ .

20. The range of a list of integers is 20, and the median is 17.

What is the smallest possible number of integers in the list?

- A 1      B 2      C 3      D 4      E 5

**SOLUTION**      **B**

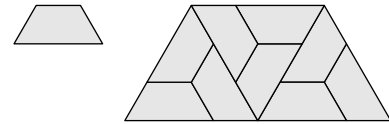
A list containing just one integer cannot have a range of 20.

We next consider whether we can find a list of two integers, say  $x, y$ , where  $x \leq y$  with a range of 20, and median 17.

If the list  $x, y$  has a median of 17, then  $\frac{1}{2}(x + y) = 17$ . If it has a range of 20, then  $y - x = 20$ , that is,  $y = x + 20$ .

Substituting  $x + 20$  for  $y$  in the equation  $\frac{1}{2}(x + y) = 17$  gives  $\frac{1}{2}(x + x + 20) = 17$ . It follows that  $x + 10 = 17$ . Hence  $x = 7$  and so  $y = 27$ . Thus 7, 27 is a list of two integers with range 20 and median 17. Therefore the smallest possible number of integers in such a list is 2.

**21.** The small trapezium on the right has three equal sides and angles of  $60^\circ$  and  $120^\circ$ . Nine copies of this trapezium can be placed together to make a larger version of it, as shown.



The larger trapezium has perimeter 18 cm.

What is the perimeter of the smaller trapezium?

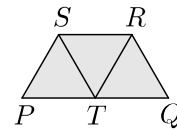
- A 2 cm      B 4 cm      C 6 cm      D 8 cm      E 9 cm

**SOLUTION**

**C**

We consider first the small trapezium.

We have labelled its vertices  $P$ ,  $Q$ ,  $R$  and  $S$ , as shown in the figure.



We let  $T$  be the point on  $PQ$  such that  $PT = SP$ .

We are told in the question that the angle  $\angle TPS$  between the equal sides  $PT$  and  $SP$  is  $60^\circ$ . It follows that the triangle  $PTS$  is equilateral. In particular,  $TS = PT = SP$  and  $\angle PST = 60^\circ$ .

Therefore because  $\angle PSR = 120^\circ$ , it follows that  $\angle TSR = 60^\circ$ . Therefore, because  $SR = SP = TS$ , the triangle  $STR$  is equilateral.

In particular,  $TR = SR$  and  $\angle SRT = 60^\circ$ . Therefore  $TR = RQ$ , and  $\angle TRQ = 60^\circ$ . Hence the triangle  $TQR$  is also equilateral. It follows that  $TQ = RQ = SR = PT$ . Hence  $PQ = PT + TQ = SR + SR$ . Thus  $PQ = 2 \times SR = 2 \times SP = 2 \times RQ$ .

Therefore in the trapezium  $PQRS$  the side  $PQ$  is twice as long as each of the other three sides. Let the length of the three equal sides each be  $x$  cm. Then the length of  $PQ$  is  $2x$  cm and hence the perimeter of  $PQRS$  is  $(2x + 3x)$  cm =  $5x$  cm.

The perimeter of the larger trapezium is made up of 5 of the shorter sides of the small trapezium and 5 of its longer sides. Hence the perimeter of the larger trapezium is  $5 \times x$  cm +  $5 \times 2x$  cm =  $15x$ . This is three times the perimeter of the smaller trapezium. The perimeter of the larger trapezium is 18 cm. It follows that the perimeter of the smaller trapezium is 6 cm,

**FOR INVESTIGATION**

**21.1** In the above solution we have used the fact that if a triangle has two equal sides and the angle between these equal sides is  $60^\circ$ , then the triangle is equilateral. Prove that this is correct.

22. In the window of Bradley's Bicycle Bazaar there are some unicycles, some bicycles and some tricycles. Laura sees that there are seven saddles in total, thirteen wheels in total and more bicycles than tricycles.

How many unicycles are in the window?

A 1

B 2

C 3

D 4

E 5

SOLUTION

**B**

METHOD 1

A unicycle has one saddle and one wheel, a bicycle has one saddle and two wheels, and a tricycle has one saddle and three wheels.

Therefore as Laura sees seven saddles there were seven cycles in the window. She saw thirteen wheels. This is six *extra wheels*, that is, wheels greater than the number of cycles.

Each bicycle contributes one extra wheel, and each tricycle two extra wheels. Because the number of extra wheels is even, the number of bicycles must be even. If there were as many as six bicycles, there would be no tricycles. Therefore there are either two bicycles or four bicycles.

If there were two bicycles, four of the extra wheels would have been on the tricycles. Hence there would also have been two tricycles. This would contradict the information that there were more bicycles than tricycles.

We therefore deduce that there were four bicycles, and hence one tricycle. Because there were seven cycles in total, it follows that there were two unicycles.

NOTE

We have given the argument above in words, so that you may contrast it with the method below where we use algebra. We hope you will agree that the solution using algebra is shorter and easier to understand

METHOD 2

Let the numbers of unicycles, bicycles and tricycles be  $u$ ,  $b$  and  $t$ , respectively. From the information given in the question we see that  $u$ ,  $b$  and  $t$  are positive integers and that  $b > t$ .

Each of the cycles has one saddle. Unicycles have one wheel, bicycles two wheels and tricycles three wheels. Therefore, because Laura saw 7 saddles and 13 wheels, we deduce that

$$\begin{aligned} u + b + t &= 7, \\ \text{and } u + 2b + 3t &= 13. \end{aligned}$$

Subtracting the first equation from the second gives

$$b + 2t = 6.$$

Because  $b > 0$ , it follows that  $t = 1$  or  $t = 2$ . If  $t = 2$ , then  $b = 2$ , contradicting  $b > t$ . Therefore  $t = 1$ . Hence  $b = 4$ . Therefore from the equation  $u + b + t = 7$ , we deduce that  $u = 2$ .

**23.** The positive integers from 1 to 150 inclusive are placed in a 10 by 15 grid so that each cell contains exactly one integer. Then the multiples of 3 are given a red mark, the multiples of 5 are given a blue mark, and the multiples of 7 are given a green mark.

How many squares have more than 1 mark?

A 10

B 12

C 15

D 18

E 19

**SOLUTION**

**E**

The squares that receive more than one mark are those which contain integers which are divisible by at least two of the integers 3, 5 and 7.

The integers 3 and 5 are *coprime*, that is, they have no common factor other than 1. Therefore the integers that are divisible by 3 and 5 are precisely those that are divisible by  $3 \times 5$ , that is, by 15.

Every 15th integer is divisible by 15. Hence the number of integers in the range from 1 to 150 that are divisible by 15 is  $150 \div 15 = 10$ .

Similarly, the integers that are divisible by 3 and 7 are precisely those that are divisible by  $3 \times 7 = 21$ . Now  $150 \div 21 = 7\frac{1}{7}$ . Therefore the number of integers in the range from 1 to 150 that are divisible by 21 is the integer part of  $7\frac{1}{7}$ , that is, 7.

In the same way we see that the integers that are divisible by 5 and 7 are precisely those that are divisible by  $5 \times 7 = 35$ . Now  $150 \div 35 = 4\frac{2}{7}$ . Therefore the number of integers in the range from 1 to 150 that are divisible by 35 is the integer part of  $4\frac{2}{7}$ , that is, 4.

We have  $10 + 7 + 4 = 21$ .

This sum has been obtained by counting the integers in the range 1 to 150 that are divisible by 3 and 5, then those divisible by 3 and 7, and finally those divisible by 5 and 7, and then adding the three separate totals. It is important to note that in doing this, integers which are divisible by each of 3, 5 and 7 have been counted three times. We need to allow for this over-counting.

There is just one integer, namely 105, in the range from 1 to 150 that is divisible by 3, 5 and 7 and hence which has been counted three times in the sum  $10 + 7 + 4$ . To allow for this we subtract 2 from this sum, so that 105 is counted just once.

Therefore the number of integers in the range from 1 to 150 that are divisible by at least two of the integers 3, 5 and 7 is given by  $(10 + 7 + 4) - 2 = 19$ .

Therefore the number of squares in the grid that are given more than one mark is 19.

**FOR INVESTIGATION**

**23.1** Check that 105 is the only integer in the range from 1 to 150 that is divisible by 3, 5 and 7.

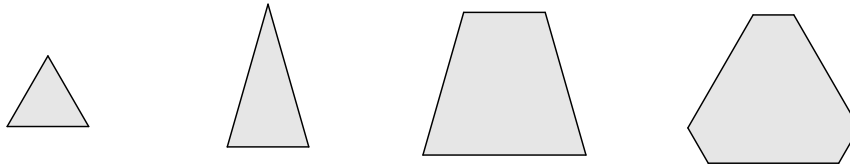
**23.2** How many integers are there in the range from 1 to 150 that are divisible by at least two of the integers 2, 5 and 11?

**23.3** How many integers are there in the range from 1 to 1000 that are divisible by at least two of the primes 2, 3, 5 and 7?



**24.** A large solid cube is cut into two pieces by a single plane cut.

How many of the following four shapes could be the shape of the cross-section formed by the cut ?



A 0

B 1

C 2

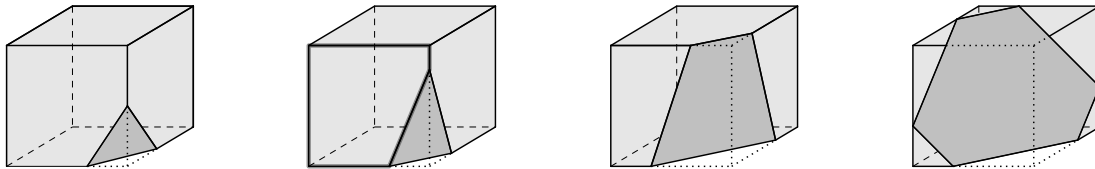
D 3

E 4

**SOLUTION**

**E**

We see from the figures below that all four shapes are possible.

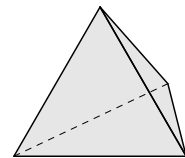


**FOR INVESTIGATION**

**24.1** Investigate which other shapes could be made by one plane cut through a solid cube.

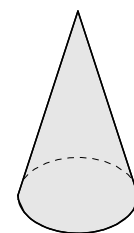
**24.2** Investigate which shapes can be made by one plane cut through a solid regular tetrahedron.

Note that a regular tetrahedron is a three-dimensional shape with four faces each of which is an equilateral triangle. It is one of the five *Platonic solids*.



**24.3** What are the Platonic solids? If you are not sure then look in a book or on the web, or ask your teacher.

**24.4** Investigate which shapes can be made by one plane cut through a solid circular cone.



**25.** The distance between Exeter and London is 175 miles. Sam left Exeter at 10:00 on Tuesday for London. Morgan left London for Exeter at 13:00 the same day. They travelled on the same road. Up to the time when they met, Sam's average speed was 25 miles per hour, and Morgan's average speed was 35 miles an hour.

At what time did Sam and Morgan meet?

- A 17:00      B 15:55      C 15:30      D 15:00      E 14:40

**SOLUTION**

**E**

**METHOD 1**

Because the answer to this question depends only on the average speeds of Sam and Morgan up to the time when they meet, we may assume that Sam's average speed was 25 miles an hour from 10:00 to 13:00 and also from 13:00 up to the time when they meet.

It follows that by 13:00 Sam has gone 75 miles and is therefore at that time 100 miles away from London. As soon as Morgan sets off they are approaching each other at a combined average speed of  $25 + 35$  miles an hour, that is, at 60 miles an hour. Therefore it takes a further  $\frac{100}{60}$  hours until they meet. Because

$$\frac{100}{60} \text{ hours} = 1\frac{40}{60} \text{ hours} = 1 \text{ hour and } 40 \text{ minutes,}$$

it follows that they meet 1 hour and 40 minutes after 13:00. Hence they meet at 14:40.

**METHOD 2**

Suppose that Sam and Morgan meet  $x$  hours after Sam sets off at 10:00.

In these  $x$  hours Sam has gone  $25x$  miles, because his average speed is 25 miles an hour.

Because Morgan sets off 3 hours after Sam, when they meet he has been travelling for  $x - 3$  hours at an average speed of 35 miles an hour. Therefore in this time he has gone  $35(x - 3)$  miles.

They start off 175 miles apart. They meet when this is the total distance they have gone between them. Therefore

$$25x + 35(x - 3) = 175.$$

It follows that

$$25x + 35x - 105 = 175,$$

and hence

$$\begin{aligned} 60x &= 175 + 105 \\ &= 280. \end{aligned}$$

Therefore

$$\begin{aligned} x &= \frac{280}{60} \\ &= 4 + \frac{40}{60}. \end{aligned}$$

Therefore they meet 4 hours and 40 minutes after 10:00. Hence they meet at 14:40.