BMOS Mentoring Scheme 2013 – 2014 (Intermediate Level)
Sheet 8 – Example Solutions and Comments

Note that these are only examples of solutions: there are several ways of doing (at least some of) these questions.

1. I need to arrange the seating of ten people around a circular dining table. In how many ways can I do this? (Two arrangements should be considered to be the same if one is a reflection or rotation of the other.)

   Answer: 181400.

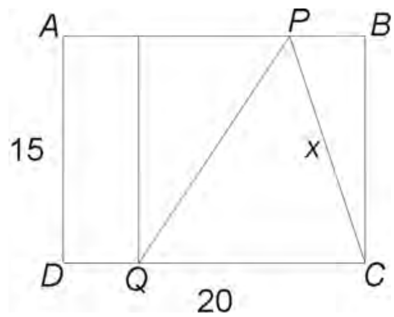
   Pick a seat for the first person. Now fill up the seats working round the table. There are 9 possible people for the first empty seat, 8 for the second, and so on until we reach the 1 person left for the last seat. This gives $9 \times 8 \times \ldots \times 2 \times 1 = 9! = 362880$ possibilities. But this counts each arrangement and its reflection separately, so we must divide by 2, which gives 181440.

   Note that by fixing the seat for the first person we have allowed for the fact that rotations of one arrangement count as the same. We could instead have observed that there are 10! arrangements of the people, but since for any one arrangement there are 10 rotations all of which count as the same arrangement we must divide by 10. (Of course, we still have to divide by 2 to allow for reflections.)

2. A rectangular piece of paper $ABCD$ measures 15cm by 20cm; the side AB is longer than the side BC. The paper is folded so that A folds exactly onto the opposite corner C. Calculate the length of the crease.

   Answer: 18.75cm.

   We have the following diagram. P and Q are the end points of the crease. If you actually fold



   a piece of paper in this way, it becomes very clear that $AP = PC$; call this distance x. Now $PB = 20 - x$, so we may apply Pythagoras' Theorem to the triangle PBC to get an equation in x. We have $x^2 = 15^2 + (20 - x)^2$, so $40x = 625$, so $x = 125/8$.

   We can use this and Pythagoras' Theorem again. Draw the perpendicular from Q to AB as shown. This gives another right-angled triangle, this time with the crease as the hypotenuse, one of the other sides having length 15cm and the other (since $DQ = 20 - x$) having length $x - (20 - x) = 2x - 20$cm.

So the length of the crease is

$$\sqrt{15^2 + (2x - 20)^2} = \sqrt{225 + \frac{15625}{16} - 1250 + 400} = \sqrt{\frac{5625}{16}} = \frac{75}{4} = 18.75\text{cm}.$$

3. Find all positive integers x that simultaneously satisfy

$$\begin{aligned} x &\equiv 1 \mod 2 \\ x^2 &\equiv 1 \mod 3 \\ x^3 &\equiv 1 \mod 5. \end{aligned}$$

Answer: All $x = 30l + 1$ and $x = 30l + 11$ for l a non-negative integer.

The condition $x \equiv 1 \mod 2$ tells us that x must be odd.

The condition $x^2 \equiv 1 \mod 3$ tells us that x must be congruent to 1 or 2 mod 3 (if $x \equiv 0 \mod 3$ then $x^2 \equiv 0 \mod 3$ so this would not do).

The condition $x^3 \equiv 1 \mod 5$ tells us that $x \equiv 1 \mod 5$ (test the other values of x mod 5 and you will see that they do not work).

Case 1 $x \equiv 1 \mod 3$. Then $x - 1$ is divisible by both 3 and 5, so is divisible by 15. So $x = 15k + 1$ for some integer $k \geq 0$. But x is odd, so we must have k even. So $x = 30l + 1$ for integers $l \geq 0$ — and these are all possible (checking in the original equations).

Case 2 $x \equiv 2 \mod 3$. We have $x \equiv 1 \mod 5$, so $x = 5k + 1$ for some integer $k \geq 0$. Now $5k + 1 \equiv 2 \mod 3$, so $2k \equiv 1 \mod 3$, so, multiplying by 2, $k \equiv 2 \mod 3$. That is, $k = 3l + 2$ for some l. So $x = 5(3l + 2) + 1 = 15l + 11$ for some integer $l \geq 0$. But x is odd, so we must have l even. So $x = 30m + 11$ for some integer $m \geq 0$ — and these are all possible (checking in the original equation).

You might like to find out about the Chinese Remainder Theorem, which is very relevant to this.

4. The set $E = \{1, 2, 3, \ldots, 2005, 2006\}$ is split into two parts, A and B:

- A consists of the numbers x in E such that the sum of the digits of x is odd;
- B consists of the numbers x in E such that the sum of the digits of x is even.

If a is the sum of the numbers in A and b is the sum of the numbers in B, what is the value of $b - a$?

Answer: 2003.

Let's think about the contribution to $b - a$ made by the first nineteen numbers in E.

1, 3, 5, 7, 9, 10, 12, 14, 16 and 18 belong to A.

2, 4, 6, 8, 11, 13, 15, 17 and 19 belong to B.

So the contribution to $b - a$ from these numbers is 0. An easy way to see this (apart from adding up the numbers!) is to pretend that 0 belongs to B (after all, this does not change b). Now each of the first five members of B is one less than the corresponding member of A, and vice versa for the second five. So these precisely cancel out.

It is not hard to see that this pattern repeats in each block of 20. So the contribution to $b - a$ from the numbers up to 1999 is 0. Now 2000, 2002, 2004 and 2006 belong to B, and 2001, 2003 and 2005 belong to A. So $b - a = 2000 + 2002 + 2004 + 2006 - 2001 - 2003 - 2005 = 2003$.

5. Find all quadruples (a, b, c, d) of positive whole numbers such that $a \leq b \leq c \leq d$ and $a + b + c + d = ab + cd$.

Answer: $(2, 2, 2, 2)$, $(1, 1, 2, 3)$, $(1, 2, 2, 3)$.

We can rearrange $a + b + c + d = ab + cd$ as $(a - 1)(b - 1) = 2 - (c - 1)(d - 1)$. Since a and b are positive whole numbers, $(a-1)(b-1) \geq 0$, so $(c-1)(d-1) \leq 2$. But $(c-1)(d-1)$ is a non-negative integer too, so $(c - 1)(d - 1)$ must be one of 0, 1 and 2.

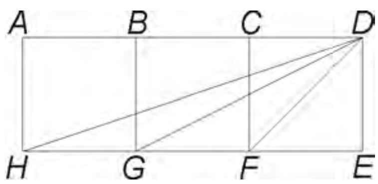
Case 1 $(c - 1)(d - 1) = 0$. This gives $(a - 1)(b - 1) = 2$. But $a \leq b \leq c \leq d$ so $(a - 1)(b - 1) \leq (c - 1)(d - 1)$. So this case is not possible.

Case 2 $(c - 1)(d - 1) = 1$. This gives $(a - 1)(b - 1) = 1$. So we must have $a - 1 = 1$, $b - 1 = 1$, $c - 1 = 1$ and $d - 1 = 1$, so $a = b = c = d = 2$ (and checking back this meets all the requirements).

Case 3 $(c-1)(d-1) = 2$. This gives $(a-1)(b-1) = 0$. Since $a \leq b$, we must have $a = 1$ (if $a > 1$ then $b > 1$ and $(a - 1)(b - 1) > 0$), but b could be any positive integer. Also, $(c - 1)(d - 1) = 2$ means that we must have $c-1 = 1$ and $d-1 = 2$, so $c = 2$ and $d = 3$. But $b \leq c$, so $b = 1$ or $b = 2$. Thus we have $(1, 1, 2, 3)$ and $(1, 2, 2, 3)$ (and both of these meet the requirements).

6. Suppose that we have three identical squares lined up next to each other to form a 1×3 rectangle. Label the eight distinct corners of the squares A, B, C, D, E, F, G and H (labelling clockwise so that A, B, C and D are in line, as are E, F, G and H). Prove that $\angle DHE + \angle DGE = \angle DFE$.

We have the following diagram:



We shall give two solutions here.

Solution 1 We shall show that triangle DHF is similar to triangle GDF.

We have $\frac{HF}{DF} = \frac{2}{\sqrt{2}} = \sqrt{2}$ and $\frac{FD}{FG} = \frac{\sqrt{2}}{1} = \sqrt{2}$, and the angle at F is the same in both triangles. So they are similar.

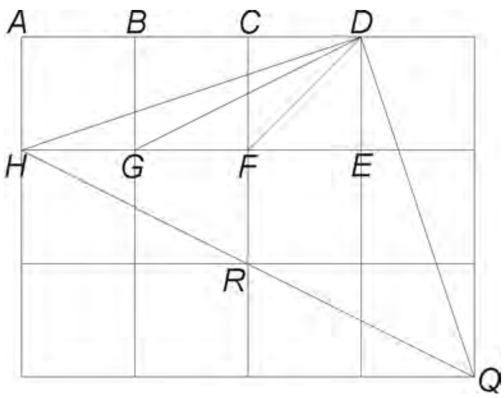
Notice that triangle DEF is isosceles with a right angle at E, so $\angle DFE = 45° = \angle ADF$.

Using properties of parallel lines (for example), we see that $\angle ADH = \angle DHE$.

Since triangle DHF is similar to triangle GDF, $\angle HDF = \angle DGF$.

But $\angle ADH + \angle HDF = 45°$, so $\angle DHE + \angle DGE = 45° = \angle DFE$.

Solution 2 We add some extra squares and two extra points to get:



Now $\angle HDQ = 90°$ and $HD = DQ$ so the triangle HDQ is isosceles, so $\angle DHQ = 45°$.

But $\angle DFE = 45°$ so $\angle DHQ = \angle DFE$. But $\angle DHQ = \angle DHE + \angle EHQ = \angle DHE + \angle EHR = \angle DHE + \angle DGE$. Thus $\angle DHE + \angle DGE = \angle DFE$.

7. A *unit fraction* is one of the form $\frac{1}{a}$ for an integer a. For any positive integer n, find a simple way to count the number of ways of writing $\frac{1}{n}$ as a sum of two positive unit fractions. Count $\frac{1}{a}+\frac{1}{b}$ and $\frac{1}{b}+\frac{1}{a}$ as different ways.
[Hint: You may express your answer in terms of the number of factors of some number that you should find.]

Answer: The number of positive factors of n^2.

We have $\frac{1}{n}=\frac{1}{a}+\frac{1}{b}$ if and only if $ab=na+nb$. We can factorise this as $(a-n)(b-n)=n^2$. So for each pair of positive factors (k,l) of n^2 (that is, $kl=n^2$), we have a pair (a,b) such that $\frac{1}{n}=\frac{1}{a}+\frac{1}{b}$ (set $a=k+n$, $b=l+n$), and for each pair (a,b) we can get back the pair of factors. So the number of ways is precisely the number of factors of n^2.

8. Six people are at a party. Any two either know each other or do not know each other. Show that there must be a group of three people all of whom know each other or all of whom do not know each other.

We shall represent the situation by a *graph*, that is, by a collection of points (vertices) and lines joining points (edges). (Note: “vertices” is plural; the singular is “vertex”.) Each point represents a person. We colour an edge red if the people at either end know each other, and blue if they do not. You should draw out the diagrams yourself. Looking for a group of three people all of whom know each other or all of whom do not know each other is the same as looking for a red triangle or a blue triangle. We shall build up the diagram in such a way that we see that we must have a triangle of a single colour.

Pick a vertex. There are five edges coming out of it, so there must be at least three of the same colour. We may as well assume that this colour is red (otherwise we could switch the names “red” and “blue”). Think about the other ends of these three edges. If an edge joining any two of these is red, then we get a red triangle (including the original vertex). But if none of them is red then we have a blue triangle. Either way we have a triangle of a single colour.

It is not too hard to show that if we only have five vertices (people) then we can avoid a triangle of a single colour (try it!).

This is traditionally the first problem in an area called Ramsey Theory. It is known that if there 18 people at the party then there is a group of 4, all of whom know each other or all of whom do not know each other, and that this is not true if there are only 17 people present. It is not known how many people are needed to guarantee a group of five!