

BMOS Mentoring Scheme

Intermediate Level 2013-14

Sheet 8

These questions are not necessarily in order of difficulty, and you do not have to attempt them in order.

1. I need to arrange the seating of ten people around a circular dining table. In how many ways can I do this? (Two arrangements should be considered to be the same if one is a reflection or rotation of the other.)
2. A rectangular piece of paper $ABCD$ measures 15cm by 20cm; the side AB is longer than the side BC . The paper is folded so that A folds exactly onto the opposite corner C . Calculate the length of the crease.
3. Find all positive integers x that simultaneously satisfy

$$\begin{aligned}x &\equiv 1 \pmod{2} \\x^2 &\equiv 1 \pmod{3} \\x^3 &\equiv 1 \pmod{5}.\end{aligned}$$

4. The set $E = \{1, 2, 3, \dots, 2005, 2006\}$ is split into two parts, A and B :
 - A consists of the numbers x in E such that the sum of the digits of x is odd;
 - B consists of the numbers x in E such that the sum of the digits of x is even.If a is the sum of the numbers in A and b is the sum of the numbers in B , what is the value of $b - a$?
5. Find all quadruples (a, b, c, d) of positive whole numbers such that $a \leq b \leq c \leq d$ and $a + b + c + d = ab + cd$.
6. Suppose that we have three identical squares lined up next to each other to form a 1×3 rectangle. Label the eight distinct corners of the squares A, B, C, D, E, F, G and H (labelling clockwise so that A, B, C and D are in line, as are E, F, G and H). Prove that $\angle DHE + \angle DGE = \angle DFE$.
7. A *unit fraction* is one of the form $\frac{1}{a}$ for an integer a . For any positive integer n , find a simple way to count the number of ways of writing $\frac{1}{n}$ as a sum of two positive unit fractions. Count $\frac{1}{a} + \frac{1}{b}$ and $\frac{1}{b} + \frac{1}{a}$ as different ways.
[Hint: You may express your answer in terms of the number of factors of some number that you should find.]
8. Six people are at a party. Any two either know each other or do not know each other. Show that there must be a group of three people all of whom know each other or all of whom do not know each other.