BMOS Mentoring Scheme 2013 – 2014 (Intermediate Level)
Sheet 7 – Example Solutions and Comments

Note that these are only examples of solutions: there are several ways of doing (at least some of) these questions.

1. Show that a 3-digit number 'abc' is divisible by 7 if and only if $2a + 3b + c$ is divisible by 7.

   Our number 'abc' is $100a + 10b + c$. We can write this as $100a + 10b + c = 98a + 7b + 2a + 3b + c = 7(14a + b) + 2a + 3b + c$. So our number is divisible by 7 if and only if $2a + 3b + c$ is.

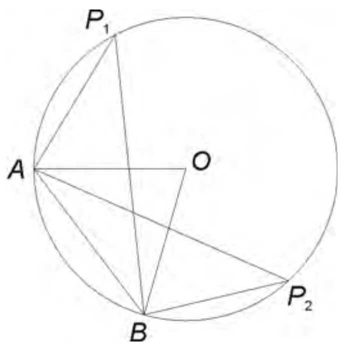
2. In this question the only circle theorem that you may assume is that proved in Question 7 on the March sheet, that is, that "the angle at the centre is twice the angle at the circumference".

   (i) Show that "angles in the same segment are equal". That is, if A and B are distinct points on the circumference of a circle, then $\angle AP_1B = \angle AP_2B$ whenever P_1 and P_2 are points on the circumference in the same segment.

   (ii) Show that opposite angles in a cyclic quadrilateral add up to 180°. That is, if $ABCD$ is a cyclic quadrilateral, then $\angle ABC + \angle CDA = 180°$.

   [A *cyclic quadrilateral* is a quadrilateral such that all four vertices lie on a circle.]

   (i) We have the following diagram, and wish to show that $\angle AP_1B = \angle AP_2B$.

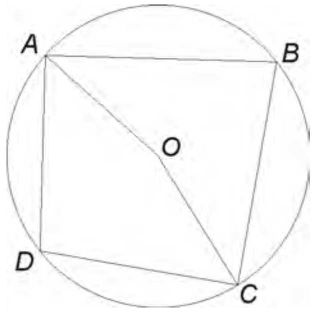
   

   But $\angle AP_1B = \frac{1}{2}\angle AOB$ and $\angle AP_2B = \frac{1}{2}\angle AOB$, so $\angle AP_1B = \angle AP_2B$.

   (ii) We have the following diagram, and wish to show that $\angle ABC + \angle CDA = 180°$.

   We know that $\angle ABC$ is half of the reflex angle at the centre of the circle, and that $\angle CDA$ is half of the other angle at the centre. Since the angles at the centre sum to 360°, we have $2\angle ABC + 2\angle CDA = 360°$, so $\angle ABC + \angle CDA = 180°$.

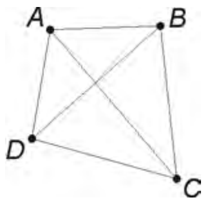
   In fact, the converse of this result is true. That is, if $ABCD$ is a quadrilateral in which opposite angles sum to 180°, then $ABCD$ is a cyclic quadrilateral. If you have time, try to prove this.



3. Is it possible to find four points in the plane such that the six distances between them are 1cm, 2cm, 3cm, 4cm, 5cm and 6cm?

Answer: No.

Suppose that it is possible, say in the configuration shown below (note that I have not yet specified which lengths are which — in particular, the diagram is not drawn to scale at all!).



We know that one of these lengths is 6cm. We shall consider several possibilities; I strongly suggest that you get a pencil and paper and draw them out for yourself.

Suppose first that it is AB. Then ABC and ABD are both triangles having a side of length 6cm. It is a property of triangles that the sum of any two sides is greater than the third side (this is called the *triangle inequality*!). You might like to think about this if you have not come across it before. We thus know that the length of AC plus the length of BC is greater than 6cm, and that the length of AD plus the length of BD is greater than 6cm. But of the remaining lengths, the only way this is possible is using $2 + 5 = 7$ and $3 + 4 = 7$. We may as well assume that AD has length 2cm and BD length 5cm. Now we only have one length left, so CD must have length 1cm. Consider the triangle BCD. CD has length 1cm and BC length 3cm or 4cm. Either way, the sum of these is not bigger than 5cm, the length of BD, violating the triangle inequality for the triangle BCD. So AB cannot have length 6cm. In the same way, none of BC, CD and DA has length 6cm.

Now suppose that AC has length 6cm. As before, we have two triangles with a side of length 6cm (namely ACD and ABC), so we can draw the same conclusions about their side lengths. We may as well assume that AB has length 2cm and BC 5cm (and that AD and CD have lengths 3cm and 4cm or vice versa). Again, we only have one length left; this time BD must have length 1cm. We can use exactly the same argument as before: think about the triangle BCD. BD has length 1cm and CD length 3cm or 4cm; either way, the sum of these is not bigger than 5cm, the length of BC. So this is not possible either. In the same way, BD cannot have length 6cm.

So such a configuration cannot exist.

If one were a little more careful, one could show that showing that AB cannot have length 6cm is enough to show that no edge can have length 6cm (after all, we used exactly the same argument in the second case). I've phrased it this way to try to make it absolutely clear that we're not cheating!

4. For how many values of n between 2001 and 2100 (inclusive) is $1^n + 2^n + 3^n + 4^n$ divisible by 5?

Answer: 75.

We shall use modular arithmetic. Notice that $3 \equiv -2 \bmod 5$ and $4 \equiv -1 \bmod 5$.

So $1^n + 2^n + 3^n + 4^n \equiv 1^n + 2^n + (-2)^n + (-1)^n \equiv (1+2^n)(1+(-1)^n) \bmod 5$.

If n is odd, then $(-1)^n = -1$, and so $1^n + 2^n + 3^n + 4^n \equiv 0 \bmod 5$. That is, $1^n + 2^n + 3^n + 4^n$ is divisible by 5.

If $n \equiv 2 \bmod 4$, then $n = 4m + 2$ for some integer m and $(-1)^n = 1$. Then $1^n + 2^n + 3^n + 4^n \equiv 2(1 + 2^{4m+2}) \bmod 5$. But $2^{4m+2} = 4^{2m+1} \equiv (-1)^{2m+1} \equiv -1 \bmod 5$, so $1^n + 2^n + 3^n + 4^n \equiv 0 \bmod 5$. That is, $1^n + 2^n + 3^n + 4^n$ is divisible by 5.

If $n \equiv 0 \bmod 4$, then $n = 4m$ for some integer m and $(-1)^n = 1$. Then $2^{4m} = 4^{2m} \equiv (-1)^{2m} \equiv 1 \bmod 5$, so $1^n + 2^n + 3^n + 4^n \equiv 4 \bmod 5$, so $1^n + 2^n + 3^n + 4^n$ is not divisible by 5.

So we have shown that $1^n + 2^n + 3^n + 4^n$ is divisible by 5 exactly when n is not divisible by 4, and there are 75 such values of n between 2001 and 2100 inclusive.

5. Take a sheet of paper. You are allowed to divide it into 8 parts or into 12 parts. Now you are allowed to divide each part into 8 parts or 12 parts (or you can leave it undivided), and so on. Is it possible to do this (dividing parts as often as necessary, but only ever into 8 parts or 12 parts) in such a way that the original piece of paper is divided into

   (i) 60 parts?

   (ii) 61 parts?

   (iii) 2002 parts?

Answer: (i) No. (ii) Yes. (iii) Yes.

We start with one part. If we divide a part into 8 parts, then we add 7 parts to the total. Similarly, if we divide it into 12 parts then we add 11 parts to the total.

(i) Suppose that it is possible, dividing m parts into 8 and n parts into 12. Then we must have $60 = 1 + 7m + 11n$, so $7m + 11n = 59$. Consider this mod 7: $4n \equiv 3 \bmod 7$. Multiply both sides by 2 to see that $8n \equiv n \equiv 6 \bmod 7$. Since n is non-negative, we find that $n \geq 6$. But then $7m + 11n \geq 66$, so 60 cannot be possible.

(ii) As in (i), it is possible if and only if there are non-negative integers m and n such that $7m + 11n = 60$. Now $4n \equiv 4 \bmod 7$, so, multiplying both sides by 2, we get $8n \equiv n \equiv 1 \bmod 7$. If $n \geq 8$ then $7m + 11n \geq 88$, so we must have $n = 1$. Putting this into $7m + 11n = 60$, we get $7m = 49$, so $m = 7$. So this is possible. For example, divide the original sheet of paper into 12, and then divide 7 of these regions into 7 and leave the other 5 alone.

(iii) Again, it is possible if and only if there are non-negative integers m and n such that $7m+11n = 2001$. We get $4n \equiv 6 \bmod 7$, so, multiplying both sides by 2, we get $n \equiv 5 \bmod 7$. We can write this as $n = 7n' + 5$ for some non-negative integer n'. Also, $7m \equiv 10 \bmod 11$, so, multiplying both sides by 8, we get $56m \equiv m \equiv 3 \bmod 11$. We can write this as $m = 11m' + 3$ for some non-negative integer m'. Substitute these expressions for m' and n' into the original equation, to get $7(11m' + 3) + 11(7n' + 5) = 2001$. Rearranging, we get $77(m' + n') = 1925$, so $m' + n' = 25$. Let's try picking $m' = 0$ and $n' = 25$. Then we get $m = 3$ and $n = 180$, and certainly $7m + 11n = 2001$. So one way of doing it is to divide the original sheet of paper into 8, then divide two of these parts into 8, and then keep picking parts to divide into 12 until we have divided 180 parts into 12. There are lots of other ways of doing this, corresponding to all the other solutions to $m' + n' = 25$.

Of course, for the latter two parts it is enough just to give values of m and n that work. I have included the rest so that you can see one way in which these values could be found (without a computer!).

6. In how many ways is it possible to choose four different positive integers such that the sum of their reciprocals is a positive integer?

Answer: 168.

Label the different positive integers a, b, c and d. Let us for now assume that $a < b < c < d$; we shall allow for the different rearrangements later. So $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$ is an integer, and $a \geq 1$, $b \geq 2$ (since $b > a$), $c \geq 3$ and $d \geq 4$. Notice that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} < 3$, so the sum of the reciprocals is 1 or 2 (if it is an integer).

If $a \geq 3$, then $b \geq 4$, $c \geq 5$, and $d \geq 6$, and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} < 1$, which is not allowed. So $a = 1$ or $a = 2$. We shall consider these separately.

Case 1 $a = 1$. Then $\frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 1$. If $b \geq 3$, then $c \geq 4$ and $d \geq 5$, and $\frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq \frac{1}{3} + \frac{1}{4} + \frac{1}{5} < 1$. So $b = 2$ and $\frac{1}{c} + \frac{1}{d} = \frac{1}{2}$.

If $c \geq 4$, then $d \geq 5$ and $\frac{1}{c} + \frac{1}{d} < \frac{1}{2}$. So $c = 3$ and $d = 6$. We get the solution $(a, b, c, d) = (1, 2, 3, 6)$.

Case 2 $a = 2$. Then $\frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{1}{2}$ or $\frac{3}{2}$. But certainly $b \geq 3$, $c \geq 4$ and $d \geq 5$, so $\frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq \frac{1}{3} + \frac{1}{4} + \frac{1}{5} < 1$. So we are looking for b, c, d such that $\frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{1}{2}$.

If $b \geq 6$, then $c \geq 7$ and $d \geq 8$. So $\frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq \frac{1}{6} + \frac{1}{7} + \frac{1}{8} < \frac{1}{2}$. So $3 \leq b \leq 5$.

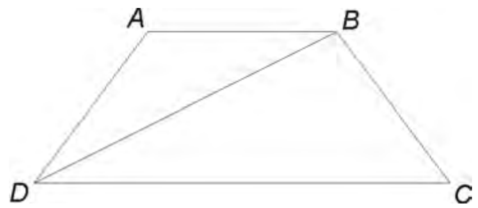
If $b = 3$, then we want $\frac{1}{c} + \frac{1}{d} = \frac{1}{6}$. Since $c < d$, $\frac{1}{c} > \frac{1}{d}$, so certainly $\frac{1}{c}$ is more than half of the sum, that is, $\frac{1}{c} > \frac{1}{12}$. So $c < 12$. Also, $\frac{1}{c} < \frac{1}{6}$, so $c > 6$. Now $d = \frac{6c}{c-6}$, and we can just check each value of c from 7 to 11 inclusive. We get the following solutions: $(2, 3, 7, 42)$, $(2, 3, 8, 24)$, $(2, 3, 9, 18)$ and $(2, 3, 10, 15)$.

If $b = 4$, then we want $\frac{1}{c} + \frac{1}{d} = \frac{1}{4}$. Following the approach of the previous paragraph, we find that $\frac{1}{c} > \frac{1}{8}$ so $c < 8$ and $\frac{1}{c} < \frac{1}{4}$ so $c > 4$. This time, $d = \frac{4c}{c-4}$. Checking the values of c from 5 to 7 inclusive, we get the solutions $(2, 4, 5, 20)$ and $(2, 4, 6, 12)$.

If $b = 5$, then we want $\frac{1}{c} + \frac{1}{d} = \frac{3}{10}$. This time we get $\frac{1}{c} > \frac{3}{20}$, so $c < \frac{20}{3} < 7$. Also, $c > b$ so $c > 5$. We have $d = \frac{10c}{3c-10}$. Checking $c = 6$, we find that this does not give a solution.

So the possible values of (a, b, c, d) with $a < b < c < d$ are $(1, 2, 3, 6)$, $(2, 3, 7, 42)$, $(2, 3, 8, 24)$, $(2, 3, 9, 18)$, $(2, 3, 10, 15)$, $(2, 4, 5, 20)$ and $(2, 4, 6, 12)$. But for each quadruple there are actually $4!$ different solutions (just reordering the values), so there are $7 \times 24 = 168$ solutions.

7. The following diagram is not to scale! Sides AB, BC and DA in this quadrilateral all have the same length. AB is parallel to DC. BC is not parallel to AD. Triangles ABD and BCD are isosceles.



Find the angles of this quadrilateral.

Answer: $\angle DAB = \angle ABC = 108°$, $\angle BCD = \angle CDA = 72°$.

Let $\theta = \angle ABD$. We know that $|AB| = |AD|$, so $\angle ADB = \theta$ and $\angle DAB = 180° - 2\theta$. Since AB and DC are parallel, $\angle BDC = \angle ABD = \theta$.

We know that BCD is an isosceles triangle, but we must establish which two sides have the same length. If BC has the same length as BD, then ABD is an equilateral triangle. So we should have $\theta = 60°$. Then BCD is also equilateral, and so CD has the same length as AB. But then BC and AD are parallel, which is not allowed. If BC has the same length as CD, then again we find that BC is parallel to AD. So BD and CD must have the same lengths.

Now we see that $\angle DBC = \angle DCB = \frac{1}{2}(180^\circ - \theta)$.

Drop perpendiculars from A and from B onto the line CD. This gives two right-angled triangles, each with the same height (since AB and DC are parallel) and the same hypotenuse (since AD and BC have the same length), so $\angle ADC = \angle BCD$. Rewriting this in terms of θ, we get $2\theta = \frac{1}{2}(180^\circ - \theta)$, so $\frac{5}{2}\theta = 90^\circ$, so $\theta = 36^\circ$. It is now easy to substitute this in to find the angles of the trapezium.

8. Let x, y, z be non-negative real numbers. Prove that at least one of the numbers $x - xy$, $y - yz$, $z - zx$ is at most $1/4$.

If $x > 1$, then $z - zx = z(1 - x) < 0 \leq 1/4$ and we are done. Similarly, if $y > 1$ or $z > 1$ then we are done. So now assume that $0 \leq x\ y,\ z \leq 1$.

We shall use the AM-GM inequality (see the March solutions for details) on the six quantities x, y, z, $1 - x$, $1 - y$ and $1 - z$ (each of which is indeed non-negative).

$$\frac{x + y + z + 1 - x + 1 - y + 1 - z}{6} \geq \sqrt[6]{xyz(1 - x)(1 - y)(1 - z)}$$

Simplifying the left-hand side and raising everything to the power of 6:

$$\frac{1}{64} \geq x(1 - y)y(1 - z)z(1 - x)$$

If $x - xy$, $y - yz$ and $z - zx$ are all greater than $1/4$, then we should have $x(1-y)y(1-z)z(1-x) > \frac{1}{64}$. But this is not the case, so at least one of $x - xy$, $y - yz$ and $z - zx$ is at most $1/4$.