BMOS Mentoring Scheme

Intermediate Level 2013-14

Sheet 7

These questions are not necessarily in order of difficulty, and you do not have to attempt them in order.

1. Show that a 3-digit number 'abc' is divisible by 7 if and only if $2a + 3b + c$ is divisible by 7.
2. In this question the only circle theorem that you may assume is that proved in Question 7 on the March sheet, that is, that "the angle at the centre is twice the angle at the circumference".

   (i) Show that "angles in the same segment are equal". That is, if A and B are distinct points on the circumference of a circle, then $\angle AP_1B = \angle AP_2B$ whenever P_1 and P_2 are points on the circumference in the same segment.

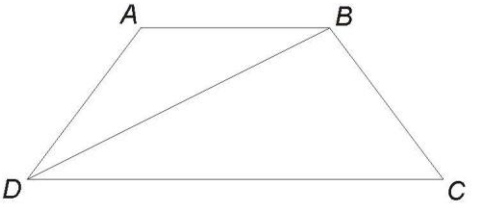
   (ii) Show that opposite angles in a cyclic quadrilateral add up to $180°$. That is, if $ABCD$ is a cyclic quadrilateral, then $\angle ABC + \angle CDA = 180°$.

   [A *cyclic quadrilateral* is a quadrilateral such that all four vertices lie on a circle.]
3. Is it possible to find four points in the plane such that the six distances between them are 1cm, 2cm, 3cm, 4cm, 5cm and 6cm?
4. For how many values of n between 2001 and 2100 (inclusive) is $1^n + 2^n + 3^n + 4^n$ divisible by 5?
5. Take a sheet of paper. You are allowed to divide it into 8 parts or into 12 parts. Now you are allowed to divide each part into 8 parts or 12 parts (or you can leave it undivided), and so on. Is it possible to do this (dividing parts as often as necessary, but only ever into 8 parts or 12 parts) in such a way that the original piece of paper is divided into

   (i) 60 parts?

   (ii) 61 parts?

   (iii) 2002 parts?
6. In how many ways is it possible to choose four different positive integers such that the sum of their reciprocals is a positive integer?
7. The following diagram is not to scale! Sides AB, BC and DA in this quadrilateral all have the same length. AB is parallel to DC. BC is not parallel to AD. Triangles ABD and BCD are isosceles.

   Find the angles of this quadrilateral.
8. Let x, y, z be non-negative real numbers. Prove that at least one of the numbers $x - xy$, $y - yz$, $z - zx$ is at most $1/4$.