BMOS Mentoring Scheme 2013 – 2014 (Intermediate Level)
Sheet 6 – Example Solutions and Comments

Note that these are only examples of solutions: there are several ways of doing (at least some of) these questions.

1. Find the value of

$$\frac{1}{\sqrt{2}+\sqrt{1}} + \frac{1}{\sqrt{3}+\sqrt{2}} + \frac{1}{\sqrt{4}+\sqrt{3}} + \ldots + \frac{1}{\sqrt{100}+\sqrt{99}}.$$

Answer: 9.

Note that $(a+b)(a-b) = a^2 - b^2$, so

$$\frac{1}{a+b} = \frac{a-b}{a^2-b^2}.$$

$$\begin{aligned}
\frac{1}{\sqrt{2}+\sqrt{1}} + \frac{1}{\sqrt{3}+\sqrt{2}} + \ldots + \frac{1}{\sqrt{100}+\sqrt{99}} &= \frac{\sqrt{2}-\sqrt{1}}{2-1} + \frac{\sqrt{3}-\sqrt{2}}{3-2} + \ldots + \frac{\sqrt{100}-\sqrt{99}}{100-99} \\
&= -\sqrt{1} + \sqrt{2} - \sqrt{2} + \sqrt{3} - \ldots - \sqrt{99} + \sqrt{100} \\
&= \sqrt{100} - \sqrt{1} \\
&= 10 - 1 \\
&= 9.
\end{aligned}$$

2. $+$ signs are written in each square of a 4×4 grid. A move consists of taking any 2×2 square and replacing each sign in it with the opposite sign.
Is it possible, after some number of moves, to get from the initial layout (all $+$ signs) to a layout in which $+$ and $-$ signs alternate like the black and white squares on a chessboard?

Answer: Yes, it is possible.

Consider the following sequence of moves (where $\overrightarrow{(a,b)}$ denotes the move using the 2×2 square with top left-hand corner (a, b))

$$\begin{array}{cccc} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{array} \quad \overrightarrow{(1,2)} \quad \begin{array}{cccc} + & - & - & + \\ + & - & - & + \\ + & + & + & + \\ + & + & + & + \end{array} \quad \overrightarrow{(1,3)} \quad \begin{array}{cccc} + & - & + & - \\ + & - & + & - \\ + & + & + & + \\ + & + & + & + \end{array} \quad \overrightarrow{(2,1)}$$

$$\begin{array}{cccc} + & - & + & - \\ - & + & + & - \\ - & - & + & + \\ + & + & + & + \end{array} \quad \overrightarrow{(2,3)} \quad \begin{array}{cccc} + & - & + & - \\ - & + & - & + \\ - & - & - & - \\ + & + & + & + \end{array} \quad \overrightarrow{(3,1)} \quad \begin{array}{cccc} + & - & + & - \\ - & + & - & + \\ + & + & - & - \\ - & - & + & + \end{array} \quad \overrightarrow{(3,2)} \quad \begin{array}{cccc} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{array}$$

3. Find all prime numbers p such that $\frac{p-1}{4}$ and $\frac{p+1}{2}$ are also prime.

Answer: $p = 13$.

Let $q = \frac{p-1}{4}$, so $p = 4q + 1$. Now p is prime if and only if $4q + 1$ is prime, $\frac{p-1}{4}$ is prime if and only if q is prime, and $\frac{p+1}{2}$ is prime if and only if $2q + 1$ is prime. So it suffices to find all primes q such that q, $2q + 1$ and $4q + 1$ are all prime.

Consider the possible values of q modulo 3.

If $q \equiv 1 \bmod 3$, then $2q + 1 \equiv 0 \bmod 3$, i.e., $2q + 1$ is divisible by 3. But $2q + 1 > 3$, so then $2q + 1$ is not prime.

If $q \equiv 2 \bmod 3$, then $4q + 1 \equiv 0 \bmod 3$, i.e., $4q + 1$ is divisible by 3. But $4q + 1 > 3$, so then $4q + 1$ is not prime.

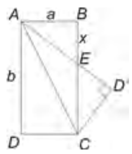
So the only possibility is $q \equiv 0 \bmod 3$. If $q > 3$, then $q \equiv 0 \bmod 3$ means that q is not prime. So we must have $q = 3$ — and this does work, since 7 and 13 are also prime.

The corresponding value of p is $4q + 1 = 13$.

4. A rectangular sheet of paper $ABCD$ is folded along the diagonal AC. The side AD in the new position intersects the side BC at point E. Find the ratio $\frac{BE}{EC}$ if $AB = a$, $AD = b$, $b > a$.

Answer: The ratio is $\frac{b^2-a^2}{b^2+a^2}$.

Label the new position of D as D', as shown.



Now $\angle DAE = \angle CED'$ (since AD and BC are parallel),

so $\angle BAE = 90° - \angle DAE = 90° - \angle CED' = \angle ECD'$.

Hence triangles ABE and $CD'E$ are similar (since their angles are the same).

Also, $AB = a$ and $CD' = CD = a$, so triangles ABE and $CD'E$ are actually congruent.

Let $BE = x$, so $EC = b - x$.

Then $b - x = EC = AE = \sqrt{a^2 + x^2}$ (by Pythagoras' Theorem),

so (squaring) $b^2 - 2bx + x^2 = a^2 + x^2$,

so $2bx = b^2 - a^2$,

so $x = \frac{b^2-a^2}{2b}$ and $b - x = \frac{b^2+a^2}{2b}$.

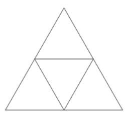
So $\frac{BE}{EC} = \frac{x}{b-x} = \frac{b^2-a^2}{b^2+a^2}$.

5. Five points are placed inside an equilateral triangle of side length 1. Show that there are two points at distance at most 1/2 apart.

This question is quite similar to Question 8 from the February sheet.

We shall use the pigeonhole principle.

Divide the triangle into four identical equilateral triangles as shown.



The smaller triangles are the pigeonholes, the points the pigeons.

By the pigeonhole principle, there is a triangle containing at least two points. But the furthest apart two points in one of the small triangles can be is $1/2$ (if they were on two vertices of the triangle).

6. m, n and k are positive integers such that $m^n|n^m$ and $n^k|k^n$. Prove that $m^k|k^m$.
[$a|b$ means that a divides b.]

This proof depends on the fact that a positive integer can be written as a product of primes in an essentially unique way. ("Essentially unique" means that the product is the same up to the order of the factors: $2 \times 3 = 3 \times 2$.) This result is called the *Fundamental Theorem of Arithmetic.*

Let p be a prime.

Let α be the largest integer such that $p^\alpha|m$ (so $p^{\alpha+1} \nmid m$ — $p^{\alpha+1}$ does not divide m).

Let β be the largest integer such that $p^\beta|n$.

Let γ be the largest integer such that $p^\gamma|k$.

Now $m^n|n^m$ tells us that $\alpha n \leq \beta m$, so $\alpha kn \leq \beta km$,

and $n^k|k^n$ tells us that $\beta k \leq \gamma n$, so $\beta km \leq \gamma mn$.

So $\alpha kn \leq \beta km \leq \gamma mn$, so $\alpha k \leq \gamma m$.

So the power of p in the prime factorisation of m^k divides k^m.

Since this is true for all primes, $m^k|k^m$.

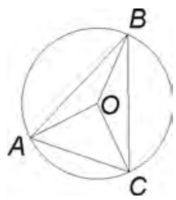
7. Let ABC be a triangle; let $a = BC$, $b = AC$, $c = AB$, and write A for the angle at A (and similarly for the other vertices). Let O denote the centre and R the radius of the circumcircle. You may assume that the angles of the triangle are all acute; this means that O lies inside the triangle. (The results are true for all triangles; this just simplifies matters slightly.)

   (i) Show that $\angle AOC = 2\angle ABC$. (You may not quote circle theorems to solve this!)

   (ii) Show that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R = \frac{abc}{2\Delta}$ where Δ is the area of the triangle ABC. (You may not quote the sine rule: the idea is that you should prove it.)

   [The circumcircle of a triangle is the circle round the triangle that just touches each of the three vertices. The circumcentre (the centre of the circumcircle) can be found by drawing the perpendicular bisectors of the three sides; they meet at a point and this is the circumcentre. $\angle ABC$ means the angle between AB and BC.]

(i) We have the situation shown in the diagram below. You are advised to draw out a large copy of this diagram for yourself and to mark on relevant things as you follow the proof (this advice applies to all geometry questions!).



Let $\alpha_1 = \angle ABO$, $\alpha_2 = \angle OBC$, $\beta = \angle AOC$. The question asks us to show that $2(\alpha_1+\alpha_2) = \beta$.
The triangle ABO is isosceles (since AO and OB are both radii of the circle), so $\angle AOB = 180° - 2\angle ABO = 180° - 2\alpha_1$.
Similarly, the triangle OBC is isosceles, so $\angle BOC = 180° - 2\alpha_2$.

The angles at O sum to $360°$: $360° = \angle AOB + \angle BOC + \angle AOC = 180° - 2\alpha_1 + 180° - 2\alpha_2 + \beta$. Rearranging, we see that $\beta = 2\alpha_1 + 2\alpha_2 = 2(\alpha_1 + \alpha_2)$.

(ii) Firstly, consider the triangle formed by taking the midpoint of BC, O and B. This is a right-angled triangle (since O, the circumcentre, lies on the perpendicular bisector of BC), and the angle at O is half of $\angle BOC = 2A$ (using part (i)) so is A. Hence $(a/2)/R = \sin A$, so $\frac{a}{2R} = \sin A$. In the same way, we find $\frac{b}{2R} = \sin B$ and $\frac{c}{2R} = \sin C$. So $2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

Now notice that the area of the triangle ABC is $\Delta = \frac{1}{2}bc \sin A = \frac{1}{2}bc \times \frac{a}{2R} = \frac{abc}{4R}$, so $2R = \frac{abc}{2\Delta}$. This uses the standard formula $\frac{1}{2}bc \sin A$ for the area of the triangle ABC. If you are not familiar with this, you should try to prove it; hopefully it is not too hard. This is a useful formula to remember, because it is often the case that one knows about the lengths of the sides but not of the height of a triangle.

If you have not come across the circumcircle before (or even if you have!), you might like to prove that the three perpendicular bisectors of a triangle always meet at a point, and that this point is equidistant from the vertices of the triangle.

8. For a, b, $c > 0$ such that $a + b + c = 1$, show

$$ab + bc + ca - abc \leq \frac{8}{27}.$$

When does equality occur?

The inequality introduced in Question 6 on the October sheet can be extended to n quantities as follows:

for $x_1, x_2, \ldots, x_n \geq 0$,

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}.$$

The quantity on the left is called the arithmetic mean (AM); the quantity on the right is the geometric mean (GM). The inequality is known as the *AM-GM inequality.* Equality occurs if and only if $x_1 = x_2 = \ldots = x_n$.

We shall apply this to the three quantities $a + b$, $b + c$ and $c + a$:

$$\frac{(a+b) + (b+c) + (c+a)}{3} \geq \sqrt[3]{(a+b)(b+c)(c+a)}.$$

So

$$\frac{2}{3}(a+b+c) = \frac{2}{3} \geq \sqrt[3]{(1-a)(1-b)(1-c)},$$

so

$$\frac{8}{27} \geq (1-a)(1-b)(1-c) = 1 - (a+b+c) + (ab+bc+ca) - abc = ab + bc + ca - abc,$$

that is,

$$ab + bc + ca - abc \leq \frac{8}{27}.$$

In order to have equality, we must have equality in the AM-GM inequality, so we must have $a = b = c = \frac{1}{3}$ (and this does indeed give equality).