BMOS Mentoring Scheme

Intermediate Level 2013-14

Sheet 6

These questions are not necessarily in order of difficulty, and you do not have to attempt them in order.

1. Find the value of
$$\frac{1}{\sqrt{2}+\sqrt{1}}+\frac{1}{\sqrt{3}+\sqrt{2}}+\frac{1}{\sqrt{4}+\sqrt{3}}+\ldots+\frac{1}{\sqrt{100}+\sqrt{99}}.$$

2. $+$ signs are written in each square of a 4×4 grid. A move consists of taking any 2×2 square and replacing each sign in it with the opposite sign.

   Is it possible, after some number of moves, to get from the initial layout (all $+$ signs) to a layout in which $+$ and $-$ signs alternate like the black and white squares on a chessboard?

3. Find all prime numbers p such that $\frac{p-1}{4}$ and $\frac{p+1}{2}$ are also prime.

4. A rectangular sheet of paper $ABCD$ is folded along the diagonal AC. The side AD in the new position intersects the side BC at point E. Find the ratio $\frac{BE}{EC}$ if $AB=a$, $AD=b$, $b>a$.

5. Five points are placed inside an equilateral triangle of side length 1. Show that there are two points at distance at most $1/2$ apart.

6. m, n and k are positive integers such that $m^n|n^m$ and $n^k|k^n$. Prove that $m^k|k^m$.

   [$a|b$ means that a divides b.]

7. Let ABC be a triangle; let $a=BC$, $b=AC$, $c=AB$, and write A for the angle at A (and similarly for the other vertices). Let O denote the centre and R the radius of the circumcircle. You may assume that the angles of the triangle are all acute; this means that O lies inside the triangle. (The results are true for all triangles; this just simplifies matters slightly.)

   (i) Show that $\angle AOC=2\angle ABC$. (You may not quote circle theorems to solve this!)

   (ii) Show that $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R=\frac{abc}{2\Delta}$ where Δ is the area of the triangle ABC. (You may not quote the sine rule: the idea is that you should prove it.)

   [The circumcircle of a triangle is the circle round the triangle that just touches each of the three vertices. The circumcentre (the centre of the circumcircle) can be found by drawing the perpendicular bisectors of the three sides; they meet at a point and this is the circumcentre. $\angle ABC$ means the angle between AB and BC.]

8. For a, b, $c>0$ such that $a+b+c=1$, show
$$ab+bc+ca-abc\leq\frac{8}{27}.$$