BMOS Mentoring Scheme 2013 – 2014 (Intermediate Level)
Sheet 5 – Example Solutions and Comments

Note that these are only examples of solutions: there are several ways of doing (at least some of) these questions.

1. I have thought of five numbers. If I add three of them in each possible way, I get 10, 14, 15, 16, 17, 17, 18, 21, 22 and 24. What are my numbers?

Answer: 2, 3, 5, 9 and 10.

Label my numbers a, b, c, d and e. We may assume that $a \leq b \leq c \leq d \leq e$ (we can choose how to label the numbers).

Now if we add three of them in each possible way and sum all ten of the resulting numbers, we get $6(a+b+c+d+e) = 10+14+15+16+17+17+18+21+22+24 = 174$, so $a+b+c+d+e = 29$.

The smallest sum of three of my numbers is $a+b+c$, so $a+b+c = 10$.

The largest is $c+d+e$, so $c+d+e = 24$.

Adding these, we see that $(a+b+c)+(c+d+e) = (a+b+c+d+e)+c = 10+24 = 34$. So $c = 5$, so $a+b = 5$ and $d+e = 19$.

The second smallest sum of three of my numbers is $a+b+d$, so $a+b+d = 14$, so $d = 9$. So $e = 10$.

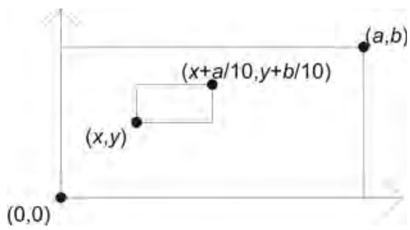
The second largest sum of three of my numbers is $b+d+e$, so $b+d+e = 22$, so $b = 3$. So $a = 2$.

So my numbers are 2, 3, 5, 9 and 10 (and these do indeed work).

2. My rectangular garden is a metres long and b metres wide. I have made a plan of it, to the scale 1:10. I place the plan flat on the ground somewhere in my garden, so that the whole map is entirely within the garden, is in the right direction and is the right way up. Is there a point in the garden that is directly underneath the point representing it on the map? Is there more than one such point?

Answer: There is exactly one such point.

We shall introduce some coordinates, to make it easier to describe points; we'll do this in the natural way (using two sides of the garden as axes). Let the bottom left-hand corner of the garden have coordinates $(0,0)$ (so the top right-hand corner has coordinates (a,b)). Let the bottom left-hand corner of the map have coordinates (x,y) (so the top right-hand corner of the map has coordinates $(x+\frac{a}{10}, y+\frac{b}{10})$).



Pick a point $(x+p, y+q)$ on the map. This point is immediately above the point in the garden that it represents if and only if $\frac{p}{x+p} = \frac{1}{10} = \frac{q}{y+q}$, if and only if $10p = x+p$ so $p = \frac{x}{9}$ and $10q = y+q$ so $q = \frac{y}{9}$. So the point $\left(\frac{10x}{9}, \frac{10y}{9}\right)$ has the required property and it is the only such.

In fact, it is clear that there is at most one such point. For if there were two such, then the distances between them on the map and in the garden would have to be the same, which is absurd.

3. The two-digit number X is equal to four times the product of the digits of the two-digit number Y. Similarly, Y is equal to four times the product of the digits of X. Find all possible such pairs X, Y.

Answer: $X = 24$, $Y = 32$ and $X = 32$, $Y = 24$.

Let X have digits a, b, so $X = 10a + b$. Similarly, let Y have digits c, d, so $Y = 10c + d$.

The conditions in the question tell us that $10a + b = 4cd$ and $10c + d = 4ab$. From the second of these, we see that $d = 4ab - 10c$ is even. So the right-hand side of the first equation is divisible by 8, so X is divisible by 8. Note that $Y \neq 0$ so $b \neq 0$, so X does not end in 0. We shall now try the various possibilities for X (namely two-digit multiples of 8 that are not also multiples of 10) to see whether any of them has a corresponding Y.

- $X = 16$. $4cd = 16$, so $cd = 4$. Also, $10c+d = 4\times1\times6 = 24$, so $d = 24-10c$. So $c(24-10c) = 4$, so $10c^2 - 24c + 4 = 0$, so $5c^2 - 12c + 2 = 0$. This has no solutions in integers.
- $X = 24$. $4cd = 24$, so $cd = 6$. Also, $10c+d = 4\times2\times4 = 32$, so $d = 32-10c$. So $c(32-10c) = 6$, so $5c^2 - 16c + 3 = 0$. This factorises as $(5c-1)(c-3) = 0$, giving $c = 3$ as the only integer solution. The corresponding value of d is 2, and the pair $X = 24$, $Y = 32$ is indeed a solution.
- $X = 32$. $4cd = 32$, so $cd = 8$. Also, $10c+d = 4\times3\times2 = 24$, so $d = 24-10c$. So $c(24-10c) = 8$, so $5c^2 - 12c + 4 = 0$. This factorises as $(5c-2)(c-2) = 0$, giving $c = 2$ as the only integer solution. The corresponding value of d is 4, and the pair $X = 32$, $Y = 24$ is indeed a solution. (Of course, we could have worked out that this would be a solution from the previous case, since X and Y can be switched.)
- $X = 48$. $4cd = 48$, so $cd = 12$. Also, $10c + d = 4 \times 4 \times 8 = 128$, so $d = 128 - 10c$. So $c(128 - 10c) = 12$, so $5c^2 - 64c + 6 = 0$. This has no solutions in integers.
- $X = 56$. $4cd = 56$, so $cd = 14$. Also, $10c + d = 4 \times 5 \times 6 = 120$, so $d = 120 - 10c$. So $c(120 - 10c) = 14$, so $5c^2 - 60c + 7 = 0$. This has no solutions in integers.
- $X = 64$. $4cd = 64$, so $cd = 16$. Also, $10c + d = 4 \times 6 \times 4 = 96$, so $d = 96 - 10c$. So $c(96 - 10c) = 16$, so $5c^2 - 48c + 8 = 0$. This has no solutions in integers.
- $X = 72$. $4cd = 72$, so $cd = 18$. Also, $10c + d = 4 \times 7 \times 2 = 56$, so $d = 56 - 10c$. So $c(56 - 10c) = 18$, so $5c^2 - 28c + 9 = 0$. This has no solutions in integers.
- $X = 88$. $4cd = 88$, so $cd = 22$. Also, $10c + d = 4 \times 8 \times 8 = 256$, so $d = 256 - 10c$. So $c(256 - 10c) = 22$, so $5c^2 - 128c + 11 = 0$. This has no solutions in integers.
- $X = 96$. $4cd = 96$, so $cd = 24$. Also, $10c + d = 4 \times 9 \times 6 = 216$, so $d = 216 - 10c$. So $c(216 - 10c) = 24$, so $5c^2 - 108c + 12 = 0$. This has no solutions in integers.

So the only possible values are $X = 24$, $Y = 32$ and $X = 32$, $Y = 24$.

4. Find, in terms of the positive real numbers p and q, the area of the quadrilateral whose vertices have coordinates (p,p), (q,p), $(2q,q)$ and $(p+q,q)$.

Answer: $(p-q)^2$.

There are two cases to consider here: $p > q$ and $p < q$ (the case $p = q$ gives a degenerate quadrilateral with area $0 = (p-q)^2$).

Case 1: $p > q$. The quadrilateral is a parallelogram with height $p - q$ and base $p - q$, so it has area $(p-q)^2$.

Case 2: $p < q$. The quadrilateral is a parallelogram with height $q - p$ and base $q - p$, so it has area $(q - p)^2$.

Finally, note that $(p - q)^2 = (q - p)^2$, so the answer is the same.

5. How many years between 2001 and 2100 inclusive have the property that dividing the year number by each of 2, 3, 4 and 5 leaves a remainder of 1?

Answer: One (namely 2041).

Let x be a year number with the required property.

Then $x - 1$ is divisible by 2, 3, 4 and 5. This happens if and only if $x - 1$ is divisible by $3 \times 4 \times 5 = 60$.

So we wish to count the numbers between 2001 and 2100 inclusive that are one more than a multiple of 60. This is the same as counting the number of multiples of 60 between 2000 and 2099 inclusive.

$33 < \frac{2000}{60} < 34$ and $34 < \frac{2099}{60} < 35$, so there is precisely one multiple of 60 in the range, namely $60 \times 34 = 2040$.

So there is only one year with the required properties, namely 2041.

6. Eight snooker players have entered a tournament. It will be a knockout tournament, with three rounds. In how many different ways can I set up the tournament? The order of the two players in each match doesn't matter. For example, Gavin vs. Jo would be the same as Jo vs. Gavin. The order of the matches is not important either: if Gavin plays Jo, then Paul plays Jenny and the two winners meet in the next round, this is the same as if Jenny plays Paul and then Gavin plays Jo and the two winners meet in the next round.

Answer: 315.

If the order of the players and matches mattered, there would be 8! ways of arranging the players (pick one of the eight players for the first slot, one of the seven remaining players for the second slot, one of the six remaining players for the third slot, and so on until left with the last player for the eighth slot). But some of these selections give the same arrangement, so we must divide by something to allow for this.

Firstly, note that the order of players in a first round match does not matter (Gavin playing Jo is the same as Jo playing Gavin). So we have counted each pair twice and there are four pairs, so we should divide by 2^4.

Next, label the pairs playing the in first round as A, B, C and D. We are told that the winners of A and B meeting in the second round is the same as if the winners of B and A meet in the second round. So we have counted this pair twice, and similarly with C and D. So we must divide by a further 2^2. Similarly, we must divide for two to allow for the fact that it does not matter in which order the semifinals occur.

So the answer is $8!/(2^4 \times 2^2 \times 2) = 8!/2^7 = 7 \times 3 \times 5 \times 3 = 315$.

7. Let x, y and z be positive real numbers such that $xyz = 1$. Show that

$$\frac{1}{1+x+xy} + \frac{1}{1+y+yz} + \frac{1}{1+z+zx} = 1.$$

We have

$$\begin{aligned}
\frac{1}{1+x+xy} + \frac{1}{1+y+yz} + \frac{1}{1+z+zx} &= \frac{1}{1+x+xy} + \frac{x}{x+xy+xyz} + \frac{xy}{xy+xyz+x^2yz} \\
&= \frac{1}{1+x+xy} + \frac{x}{1+x+xy} + \frac{xy}{1+x+xy} \\
&= \frac{1+x+xy}{1+x+xy} \\
&= 1.
\end{aligned}$$

8. Ten points are placed inside a unit square (a square with sides of length 1). Show that there is a pair of points at most $\frac{\sqrt{2}}{3}$ apart.

We shall use the pigeonhole principle.

Divide up the square into nine squares each of side length 1/3, as shown below.

These smaller squares are the pigeonholes and the points are the pigeons. There are 9 squares and 10 points, so, by the pigeonhole principle, there is a square containing at least two points. But if two points are contained in a square of side length 1/3, then they are at distance at most $\frac{\sqrt{2}}{3}$ from each other (since this is the length of the diagonal of the square).