BMOS Mentoring Scheme 2013 – 2014 (Intermediate Level)
Sheet 4 – Example Solutions and Comments

Note that these are only examples of solutions: there are several ways of doing (at least some of) these questions.

1. **Show that amongst any collection of four natural numbers there are two that leave the same remainder on division by 3.**

   There are three possible remainders on division by 3 (0, 1 and 2).

   We're going to use the pigeonhole principle (see the October solutions). We have four natural numbers in our collection (the pigeons) and three remainders on division by 3 (the pigeonholes), so, by the pigeonhole principle, two of the numbers leave the same remainder on division by 3.

2. $a_1a_2\ldots a_n$ is the decimal notation for a number a (so, for example, if $a = 135$ then $a_1 = 1$, $a_2 = 3$ and $a_3 = 5$). Prove that

   (i) a is a multiple of 2 if and only if a_n is;

   (ii) a is a multiple of 5 if and only if a_n is;

   (iii) a is a multiple of 3 if and only if $a_1 + a_2 + \ldots + a_n$ is;

   (iv) a is a multiple of 9 if and only if $a_1 + a_2 + \ldots + a_n$ is;

   (v) a is a multiple of 11 if and only if $a_1 - a_2 + a_3 - \ldots \pm a_n$ (where + and − signs alternate) is.

   It will be convenient to use the notation of modular arithmetic for this solution. Of course, this is not necessary; it just makes some arguments easier to write out (and we'll use it for the easier parts as practice).

   Note that $a = 10^{n-1}a_1 + 10^{n-2}a_2 + \ldots + 10a_{n-1} + a_n$.

   (i) $10^k \equiv 0 \bmod 2$ for $k \geq 1$ (since 10 is divisible by 2),
   so $a \equiv a_n \bmod 2$.
   So $a \equiv 0 \bmod 2$ if and only if $a_n \equiv 0 \bmod 2$,
   i.e., a is a multiple of 2 if and only if a_n is.

   (ii) $10^k \equiv 0 \bmod 5$ for $k \geq 1$ (since 10 is divisible by 5),
   so $a \equiv a_n \bmod 5$; now similar to (i).

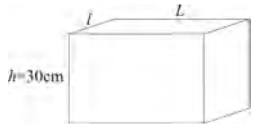
   (iii) $10^k \equiv 1 \bmod 3$ for $k \geq 1$ ($10 \equiv 1 \bmod 3$ and we can multiply congruences),
   so $a \equiv a_1 + a_2 + \ldots + a_n \bmod 3$; now similar to (i).

   (iv) $10^k \equiv 1 \bmod 9$ for $k \geq 1$,
   so $a \equiv a_1 + a_2 + \ldots + a_n \bmod 9$; now similar to (i).

(v) $10^k \equiv (-1)^k \bmod 11$ for $k \geq 0$ (check $k = 0$; $10 \equiv -1 \bmod 11$),
so $a \equiv (-1)^{n-1}a_1 + (-1)^{n-2}a_2 + \ldots - a_{n-1} + a_n \equiv (-1)^{n-1}[a_1 - a_2 + \ldots + (-1)^{n-1}a_n] \bmod 11$.
So $a \equiv 0 \bmod 11$ if and only if $(-1)^{n-1}[a_1 - a_2 + \ldots + (-1)^{n-1}a_n] \equiv 0 \bmod 11$, if and only if $a_1 - a_2 + \ldots + (-1)^{n-1}a_n \equiv 0 \bmod 11$,
i.e., a is a multiple of 11 if and only if $a_1 - a_2 + \ldots + (-1)^{n-1}a_n$ is.

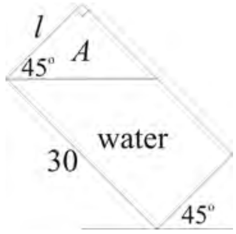
3. An aquarium is shaped like a cuboid, with width l, length L and height $h = 30$cm, where $l < 30\text{cm} < L$. It is completely full of water and is on a horizontal surface. To empty out some water, if one tilts the aquarium on one of the two longer sides of its base so that the base makes an angle of $45°$ to the horizontal, it loses one third of its water. However, if one tilts it on one of the shorter sides of its base so that the base makes an angle of $45°$ with the horizontal, it loses four fifths of its water. What is the volume of the aquarium?

Answer: 45000cm^3.



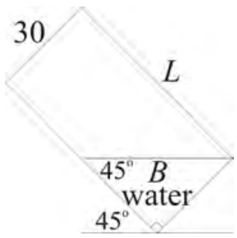
The aquarium has volume $V = 30lL\text{cm}^3$.

Tilting on a long side we have a cross-section



A is a right-angled isosceles triangle with area $\frac{1}{2} \times l \times l = \frac{l^2}{2}\text{cm}^2$. So $L \times \frac{l^2}{2} = \frac{1}{3} \times 30lL$ (the aquarium loses one third of its contents this way), so $\frac{l}{2} = 10$, so $l = 20$cm.

Tilting on a short side we have cross section



B is a right-angled isosceles triangle with area $\frac{1}{2} \times 30 \times 30 = 450\text{cm}^2$. So $l \times 450 = \frac{1}{5} \times 30lL$ (one fifth of the water remains), so $450 = 6L$, i.e., $L = 75$cm.

So the volume of the aquarium is $30 \times 20 \times 75 = 45000\text{cm}^3$.

4. The four positive real numbers a, b, c, d are such that $a > b > c > d$ and $a + b + c + d = 1$. Is the inequality $a^2 + 3b^2 + 5c^2 + 7d^2 < 1$ always valid?

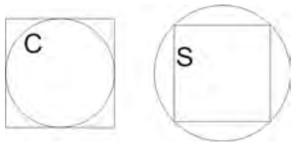
Answer: Yes.

We shall show that, under these conditions on a, b, c and d, the inequality *is* always valid. For

$$\begin{aligned} 1 = (a+b+c+d)^2 &= a^2 + b^2 + 2ab + c^2 + 2ac + 2bc + d^2 + 2ad + 2bd + 2cd \\ &> a^2 + b^2 + 2b^2 + c^2 + 2c^2 + 2c^2 + d^2 + 2d^2 + 2d^2 + 2d^2 \end{aligned}$$

(since $a > b > c > d$),
so $1 > a^2 + 3b^2 + 5c^2 + 7d^2$.

5. Let C be a circle inscribed inside a square of area 1, as shown in the left-hand diagram. Let S be a square inscribed inside a circle of area 1, as shown in the right-hand diagram. Which of C and S has the larger area?



Answer: C has the larger area.

Let the radius of C be r. The square in which C is inscribed has area 1 and hence side length 1. So $2r = 1$, so C has area $\pi \times \left(\frac{1}{2}\right)^2 = \frac{\pi}{4}$.

Let d denote the side length of S. The circle in which S is inscribed has area 1 and hence radius $\sqrt{\frac{1}{\pi}} = \frac{1}{\sqrt{\pi}}$. So the diagonal of S, which has length $d\sqrt{2}$, is $2 \times \frac{1}{\sqrt{\pi}}$. That is, $d\sqrt{2} = \frac{2}{\sqrt{\pi}}$, so $d = \sqrt{\frac{2}{\pi}}$. So S has area $d^2 = \frac{2}{\pi}$.

Now $\frac{\pi}{4} > \frac{2}{\pi}$ (it is easy to see this since $\pi^2 > 9 > 8$).

So C has a larger area.

6. Find all triples (x, y, z) of integers that satisfy the following system of equations:

$$\begin{aligned} x^3 - 4x^2 - 16x + 60 &= y \\ y^3 - 4y^2 - 16y + 60 &= z \\ z^3 - 4z^2 - 16z + 60 &= x \end{aligned}$$

Answer: $(-4, -4, -4)$, $(3, 3, 3)$, $(5, 5, 5)$.

We shall number the equations for convenience:

$$\begin{aligned} x^3 - 4x^2 - 16x + 60 &= y && (1) \\ y^3 - 4y^2 - 16y + 60 &= z && (2) \\ z^3 - 4z^2 - 16z + 60 &= x && (3) \end{aligned}$$

(1)−(2) gives $x^3 - y^3 - 4x^2 + 4y^2 - 16x + 16y = y - z$. We can factorise this as $(x-y)[(x^2 + xy + y^2) - 4(x+y) - 16] = y - z$. Since we are only interested in integers, we see that either $x = y$ (and so $y = z$) or $x - y$ divides $y - z$, which we may write as $(x-y)|(y-z)$. (This is standard notation for "divides".)

Case 1: $x = y = z$ In equation (1), this gives $x^3 - 4x^2 - 17x + 60 = 0$, which we may factorise as $(x+4)(x-3)(x-5) = 0$. So the only possibilities are $(-4, -4, -4)$, $(3, 3, 3)$ and $(5, 5, 5)$, and these are indeed solutions.

Now let's suppose that we don't have $x = y = z$.

Similarly, we find (from (2)-(3)) that $y - z$ divides $z - x$ and (from (3)-(1)) that $z - x$ divides $x - y$.

But $(x-y)|(y-z)$ and $(y-z)|(z-x)$ means $(x-y)|(z-x)$. So $x - y$ divides $z - x$ and vice versa. So $x - y = \pm(z - x)$.

Case 2: $x - y = -z + x$ so $y = z$. Now we have $x = y = z$ and we are in fact in Case 1.

Case 3: $x - y = z - x$. If $x - y = -y + z$, then, as in Case 2, we actually have $x = y = z$, so suppose that $x - y = y - z$. Now $2(x - y) = x - y + z - x = z - y = -(x - y)$, so $x = y = z$ and we are again in Case 1.

So these three are the only solutions in integers.

7. Is there a positive integer N such that writing N, N^2 and N^3 uses each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 exactly once?

Answer: No.

Suppose that N is such a positive integer. Note that N, N^2 and N^3 between them must have exactly 10 digits.

N cannot have just 1 digit: if N had 1 digit then N^2 should have at most 2 digits and N^3 at most 3, and $1 + 2 + 3 < 10$.

N cannot have 3 or more digits: if N had 3 digits then N^2 should have at least 5 digits and N^3 at least 7, and $3 + 5 + 7 > 10$.

So N has 2 digits.

How many digits can N^2 have?

Since N has 2, N^2 must have at least 3 digits.

If N^2 has at least 4 digits, then N^3 has at least 5, and $2 + 4 + 5 > 10$ so this is not allowed.

So N^2 must have 3 digits. So $N^2 < 1000$, so $N \leq 31$.

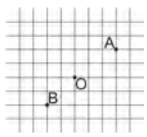
Now N^3 must have $10 - 2 - 3 = 5$ digits. So $N^3 > 9999$, so $N \geq 22$.

So $22 \leq N \leq 31$.

But, as is easily checked, none of the integers between 22 and 31 has the required property.

So no such N exists.

8. On a grid of squares of side 1, as shown below, we define the distance between two lattice points (points of the grid where lines intersect) to be the length of the shortest path between the points that follows the lines of the grid. For example, in the picture below the distance from O to A is 5, and the distance from O to B is 4.



How many lattice points are there:

(i) at distance 1 from O?

(ii) at distance 10 from O?

(iii) at distance n from O?

Answer: (i) 4; (ii) 40; (iii) $4n$.

(i) It is easy to count in this case: there are just 4.

(ii) We shall answer this by solving (iii) and setting $n = 10$.

(iii) Note that we can consider how many points there are at distance n from O in one quadrant (including one bounding axis but not both, so each is included in exactly one quadrant) and then multiply by 4, since the quadrants will contain the same number of points. So let's consider the top right-hand (positive) quadrant, with the positive part of the x-axis, but none of the y-axis.

Each point in this quadrant is uniquely determined by its coordinates, which must sum to n (the point is n away from O, so there are a total of n horizontal and vertical steps from O to n). There are exactly n such points, corresponding to x values $1, 2, \ldots, n$. (We have excluded 0 as this will be in another quadrant.) So there are n such points in this quadrant, and hence $4n$ total.