

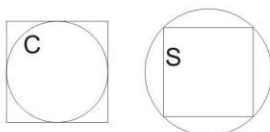
BMOS Mentoring Scheme

Intermediate Level 2013-14

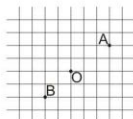
Sheet 4

These questions are not necessarily in order of difficulty, and you do not have to attempt them in order.

- Show that amongst any collection of four natural numbers there are two that leave the same remainder on division by 3.
- $a_1a_2 \dots a_n$ is the decimal notation for a number a (so, for example, if $a = 135$ then $a_1 = 1$, $a_2 = 3$ and $a_3 = 5$). Prove that
 - a is a multiple of 2 if and only if a_n is;
 - a is a multiple of 5 if and only if a_n is;
 - a is a multiple of 3 if and only if $a_1 + a_2 + \dots + a_n$ is;
 - a is a multiple of 9 if and only if $a_1 + a_2 + \dots + a_n$ is;
 - a is a multiple of 11 if and only if $a_1 - a_2 + a_3 - \dots \pm a_n$ (where $+$ and $-$ signs alternate) is.
- An aquarium is shaped like a cuboid, with width l , length L and height $h = 30\text{cm}$, where $l < 30\text{cm} < L$. It is completely full of water and is on a horizontal surface. To empty out some water, if one tilts the aquarium on one of the two longer sides of its base so that the base makes an angle of 45° to the horizontal, it loses one third of its water. However, if one tilts it on one of the shorter sides of its base so that the base makes an angle of 45° with the horizontal, it loses four fifths of its water. What is the volume of the aquarium?
- The four positive real numbers a, b, c, d are such that $a > b > c > d$ and $a + b + c + d = 1$. Is the inequality $a^2 + 3b^2 + 5c^2 + 7d^2 < 1$ always valid?
- Let C be a circle inscribed inside a square of area 1, as shown in the left-hand diagram. Let S be a square inscribed inside a circle of area 1, as shown in the right-hand diagram. Which of C and S has the larger area?



- Find all triples (x, y, z) of integers that satisfy the following system of equations:
$$\begin{aligned}x^3 - 4x^2 - 16x + 60 &= y \\y^3 - 4y^2 - 16y + 60 &= z \\z^3 - 4z^2 - 16z + 60 &= x\end{aligned}$$
- Is there a positive integer N such that writing N , N^2 and N^3 uses each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 exactly once?
- On a grid of squares of side 1, as shown below, we define the distance between two lattice points (points of the grid where lines intersect) to be the length of the shortest path between the points that follows the lines of the grid. For example, in the picture below the distance from O to A is 5, and the distance from O to B is 4.



How many lattice points are there:

- at distance 1 from O ?
- at distance 10 from O ?
- at distance n from O ?