BMOS Mentoring Scheme 2013 – 2014 (Intermediate Level)
Sheet 3 – Example Solutions and Comments

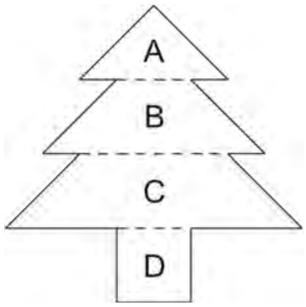
Note that these are only example solutions: there are several ways of doing (at least some of) these questions.

1. I have made a cake in the shape of a Christmas tree, as shown in the diagram below.

To work out how much icing I need, I must know the area of the surface that the icing will cover (the area of the top and the vertical faces). The depth of the cake is 10cm, and it is in the shape of a tree that is 20cm tall. The smallest triangle is 10cm wide and 5cm tall, and each pair of 'branches' is 5cm wider than the previous pair (so the cake is 20cm wide at its widest point). Each branch tapers in such a way that at the top it is 5cm narrower than the widest part of the branch above (so the indentations are 2.5cm wide). The height between branches is 5cm. The 'trunk' is a 5cm by 5cm square. Find the surface area that will be covered by icing.

Answer: $575 + 300\sqrt{2}\text{cm}^2$

Let's first work out the area of the top of the cake. We can subdivide the shape as shown below.



A is a triangle with height 5cm and base 10cm, so has area $(1/2) \times 5 \times 10 = 25\text{cm}^2$.

B is a trapezium with height 5cm, shorter parallel side of length $10 - 5 = 5$cm and longer parallel side of length $10 + 5 = 15$cm, so has area $(1/2) \times (5 + 15) \times 5 = 50\text{cm}^2$.

C is a trapezium with height 5cm, shorter parallel side of length $15 - 5 = 10$cm and longer parallel side of length $15 + 5 = 20$cm, so has area $(1/2) \times (10 + 20) \times 5 = 75\text{cm}^2$.

D is a square with height 5cm, so has area $5 \times 5 = 25\text{cm}^2$.

Hence the surface area of the top of the cake is 25+50+75+25=175cm^2.

Now we need to find the area of the vertical faces of the cake. We can imagine covering this area with a long rectangle with width the depth of the cake and length the perimeter of the shape. So we need to find the perimeter of the shape.

A sloping side of shape A has length $\sqrt{5^2+5^2} = 5\sqrt{2}$cm (by Pythagoras' Theorem), so the perimeter of the total shape that comes from A is $2.5 + 5\sqrt{2} + 5\sqrt{2} + 2.5 = 5 + 10\sqrt{2}$cm.

A sloping side of shape B has the same length ($5\sqrt{2}$cm), again by Pythagoras' Theorem, so the perimeter of the total shape that comes from B is $2.5 + 5\sqrt{2} + 5\sqrt{2} + 2.5 = 5 + 10\sqrt{2}$cm.

Similarly, a sloping side of shape C has length $5\sqrt{2}$cm, so the perimeter of the total shape that comes from C is $15 + 5\sqrt{2} + 5\sqrt{2} = 15 + 10\sqrt{2}$cm (the base is 20cm long, and 5cm of this is not included because of the trunk).

Three sides of D contribute to the perimeter, so the perimeter of the total shape that comes from D is $5 + 5 + 5 = 15$cm.

Hence the total perimeter of the shape is $5 + 10\sqrt{2} + 5 + 10\sqrt{2} + 15 + 10\sqrt{2} + 15 = 40 + 30\sqrt{2}$cm.

Now we see that the total area of the vertical faces is $(40 + 30\sqrt{2}) \times 10 = 400 + 300\sqrt{2}\text{cm}^2$.

Hence the area of the surfaces to be covered by icing is $175 + 400 + 300\sqrt{2} = 575 + 300\sqrt{2}\text{cm}^2$.

2. I pave my patio (which is rectangular, with side lengths that are whole numbers of metres) with 1m $\times$ 1m square slabs. I use brown slabs for the outside edges (including the corners), and cream for the inside stones. If I use the same number of brown slabs as cream, how big is my patio? (You should find all possibilities for the size of the patio.)

Answer: 12m$\times$5m or 8m$\times$6m.

Let a be the length and b the width of my patio (both in metres).

Then there are $2a + 2b - 4$ brown slabs (we find the perimeter of the rectangle and then subtract 4 to avoid counting the corners twice), and $ab - (2a + 2b - 4)$ cream slabs.

So $ab - (2a + 2b - 4) = 2a + 2b - 4$,

i.e., $ab - 4a - 4b + 8 = 0$.

So $(a - 4)(b - 4) = 8$ (adding 8 to both sides and factorising).

Since a and b are integers, we know that $a - 4$ and $b - 4$ are factors of 8. Also, $a - 4$, $b - 4 \geq -3$ (as a, $b \geq 1$), so we cannot include negative factors.

We consider the cases separately. Clearly we can swap a and b, so we only need to think about one way round for each pair of factors.

$a - 4 = 8$, $b - 4 = 1$ gives $a = 12$, $b = 5$.

$a - 4 = 4$, $b - 4 = 2$ gives $a = 8$, $b = 6$.

So my patio is 12m$\times$5m or 8m$\times$6m.

3. Let $\binom{n}{r}$ denote the number of ways of choosing r objects from n (see the November solutions for more details). Show that $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$.
Can you do this question without using the formula (involving factorials) for $\binom{n}{r}$?

Below are two possible proofs.

Answer 1.

Write it out in terms of factorials. We shall start with the right-hand side and manipulate it to show that it equals the left-hand side (of the equation as stated in the question).

$$\begin{aligned}\binom{n-1}{r}+\binom{n-1}{r-1} &= \frac{(n-1)!}{r!(n-r-1)!}+\frac{(n-1)!}{(r-1)!(n-r)!}\\ &= \frac{(n-r)(n-1)!}{r!(n-r)!}+\frac{r(n-1)!}{r!(n-r)!}\\ &= \frac{(n-1)![(n-r)+r]}{r!(n-r)!}\\ &= \frac{n(n-1)!}{r!(n-r)!}\\ &= \frac{n!}{r!(n-r)!}\\ &= \binom{n}{r}\end{aligned}$$

Answer 2.

Let's thing about what the quantities in the equation mean.

The left-hand side is the number of ways of choosing r objects from n. We shall show that the right-hand side is the same.

Pick a special object (for example, by painting one object blue and the others red). Now a group of r objects either contains the blue one or it does not (of course!).

Let's count the number of ways of choosing r objects including the blue one. We need to choose $r-1$ other objects, and there are $n-1$ objects to choose from. So there are $\binom{n-1}{r-1}$ ways.

Now let's count the number of ways of choosing r objects not including the blue one. We need to choose r objects, and there are $n-1$ to choose from. So there are $\binom{n-1}{r}$ ways.

Since the the total of these is the number of ways of choosing r objects from n, we have

$$\binom{n}{r}=\binom{n-1}{r}+\binom{n-1}{r-1}.$$

4. A *palindromic number* is one that reads the same when its digits are reversed, such as 2332. What is the largest palindromic six-digit number that is exactly divisible by 6?

Answer: 897798.

A general six-digit palindromic number x looks like '$abccba$' for some a, b, c between 0 and 9 ($a \neq 0$).

If x is divisible by 6, then it is certainly even. So a is even. We are looking for the largest x, so it would be best if we could take $a = 8$. Of course, it is possible that this will not work, but let's try it.

We also need x to be divisible by 3, so the sum of the digits must be divisible by 3, i.e., $2a+2b+2c$ is divisible by 3. So $16+2b+2c$ is divisible by 3. So $2b+2c$ leaves remainder 2 when divided by 3, so $b+c$ leaves remainder 1 when divided by 3. We want b as large as possible, and then c as large as possible so that $b+c$ leaves remainder 1 when divided by 3. So let's try $b = 9$. Then the largest possible c is 7. So 897798 is the largest possible (and clearly works).

5. How many zeros does 1000000! have at the end?

Answer: 249998.

We shall show that the number of zeros at the end of 1000000! is the number of times that 5 appears in the prime factorisation of 1000000! It turns out to be convenient to first work out this number.

Write $1000000! = 2^a 5^b m$ where a, b are positive integers and m is an integer not divisible by 2 or 5. So b is the number mentioned above that we wish to find.

A small piece of notation: we write $\lfloor x \rfloor$ for the greatest integer less than or equal to x, sometimes called the *integer part* of x, or the *floor function*. So $\lfloor 5.1 \rfloor = 5$, $\lfloor 5.9 \rfloor = 5$, $\lfloor 46/9 \rfloor = 5$ and $\lfloor 5 \rfloor = 5$, for example. ($\lceil x \rceil$, the *ceiling function*, is the least integer greater than or equal to x.)

Of the integers between 1 and 1000000 (inclusive), there are

- $\frac{1000000}{5} = 200000$ that are divisible by 5;
- $\frac{1000000}{25} = 40000$ that are divisible by 5^2;
- $\frac{1000000}{125} = 8000$ that are divisible by 5^3;
- $\frac{1000000}{625} = 1600$ that are divisible by 5^4;
- $\frac{1000000}{3125} = 320$ that are divisible by 5^5;
- $\frac{1000000}{15625} = 64$ that are divisible by 5^6;
- $\lfloor \frac{1000000}{78125} \rfloor = 12$ that are divisible by 5^7;
- $\lfloor \frac{1000000}{390625} \rfloor = 2$ that are divisible by 5^8;
- $\lfloor \frac{1000000}{1953125} \rfloor = 0$ that are divisible by 5^9.

Hence 1000000! is divisible by 5^b (but not by 5^{b+1}) where

$$b = 200000 + 40000 + 8000 + 1600 + 320 + 64 + 12 + 2 = 249998.$$

Half of the integers between 1 and 1000000 are even, so $a \geq 500000$. So $a > b$, so $1000000! = 2^b 5^b 2^{a-b} m = 10^b 2^{a-b} m$. But 10 does not divide $2^{a-b} m$ (as m is not divisible by 5), so 1000000! has exactly $b = 249998$ zeros at the end.

6. Find all ordered triples (x, y, z) of real numbers that satisfy the following system of equations:

$$\begin{aligned} xy &= z - x - y \\ yz &= x - y - z \\ zx &= y - x - z \end{aligned}$$

(*Ordered* triples means that we count $(1, 2, 3)$ as different from $(2, 1, 3)$ etc.: the order matters.)

Answer: $(0, 0, 0)$, $(-1, -1, -1)$, $(-2, -2, 0)$, $(-2, 0, -2)$, $(0, -2, -2)$.

We can rearrange the equations to give

$$\begin{aligned} (x+1)(y+1) &= z+1 \qquad (1) \\ (y+1)(z+1) &= x+1 \qquad (2) \\ (z+1)(x+1) &= y+1 \qquad (3) \end{aligned}$$

Multiplying (1) and (2), we get $(x+1)(y+1)^2(z+1) = (x+1)(z+1)$. Hence we have three cases: $x = -1$, $z = -1$, or $(y+1)^2 = 1$.

Case 1: $x = -1$. Then, from (1), $z = -1$, and from (3), $y = -1$. So we have the triple $(-1, -1, -1)$, which is indeed a solution.

Case 2: $z = -1$. Similarly, we get $x = -1$ and $y = -1$, so we get the same solution.

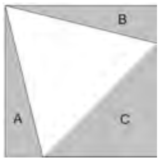
Case 3: $(y+1)^2 = 1$.

(a) $y + 1 = 1$, so $y = 0$. Then from (1) (or (2)) we get $z = 2$, and from (3) we have $(x + 1)^2 = 1$. So we have $x = 0$ or $x + 1 = -1$ (and hence $x = -2$). This gives the triples $(0, 0, 0)$ and $(-2, 0, -2)$, which are both easily seen to be solutions.

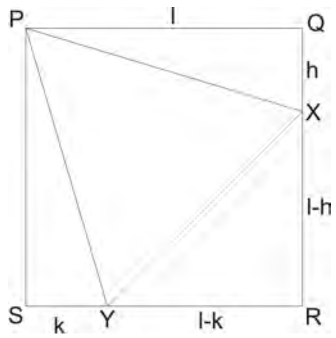
(b) $y + 1 = -1$, so $y = -2$. Then, from (1), $-(x + 1) = z + 1$. Substituting this into (3), we have $-(x + 1)^2 = -1$, so $(x + 1)^2 = 1$. As before, $x = 0$ or $x = -2$. If $x = 0$, we have $(0, -2, -2)$ (which is a solution), and if $x = -2$, we have $(-2, -2, 0)$, which is also a solution.

Of course, once we had found the solution $(-2, 0, -2)$ we could have immediately deduced that $(-2, -2, 0)$ and $(0, -2, -2)$ must also be solutions, by symmetry (permuting x, y, z gives the same set of equations), but it is important to be sure that we have found all of the solutions.

7. The diagram shows an equilateral triangle inside a square. Prove that area A + area B = area C.



Label the vertices of the square as P, Q, R, S, the vertex of the triangle on the side QR as X, and the vertex of the triangle on the side RS as Y, as shown in the diagram below. Let the square have side length l, the line segment QX length h, and the line segment SY length k.



By Pythagoras' Theorem (applied to the right-angled triangles PQX and PSY) we see that $k = h$, since $PY = PX$ (the triangle is equilateral).

To save space, we write A, B and C for the areas of the regions A, B and C respectively.

$A = \frac{1}{2}lh = B$, so it is sufficient to show that $A = \frac{1}{2}C$.

Note that $C = \frac{1}{2}(l - h)^2$.

Also, $l^2 + h^2 = 2(l - h)^2$ (by Pythagoras' Theorem applied to the right-angled triangles PQX and XRY, using the fact that $PX = XY$).

So $l^2 + h^2 = 2l^2 - 4lh + 2h^2$, so $4lh = l^2 + h^2$, so $2lh = l^2 - 2lh + h^2 = (l - h)^2$.

So $A = \frac{1}{2}lh = \frac{1}{4}(l - h)^2 = \frac{1}{2}C$.

8. Prove that for any six consecutive positive integers there is a prime number that divides exactly one of these integers.

Write our consecutive positive integers as n, $n + 1$, $n + 2$, $n + 3$, $n + 4$, $n + 5$.

Note that if an integer is greater than 5 then it divides at most one of these numbers, because if it were to divide two then it would also divide their difference, which is at most 5.

If n is not a multiple of 5, then exactly one of these six numbers is divisible by 5, so we are done. So now let us assume that n is a multiple of 5.

Then $n + 5$ is also a multiple of 5, and none of $n + 1$, $n + 2$, $n + 3$, $n + 4$ is.

If $n + 1$ is divisible by 3, then neither of $n + 2$ and $n + 3$ is. Also, one of $n + 2$, $n + 3$ must be odd (that is, not divisible by 2). Then this number is not divisible by any of 2, 3 and 5, and so is divisible by some prime $p \geq 7$ (and, as observed above, p does not divide any of the others).

If $n + 1$ is not divisible by 3, then neither is $n + 4$. One of $n + 1$, $n + 4$ is odd; then, similarly to above, it is not divisible by any of 2, 3 and 5, so is divisible by some prime $p \geq 7$ that does not divide any of the others.