BMOS Mentoring Scheme

Intermediate Level 2013-14

Sheet 3

These questions are not necessarily in order of difficulty, and you do not have to attempt them in order.

1. I have made a cake in the shape of a Christmas tree, as shown in the left-hand diagram below.

   To work out how much icing I need, I must know the area of the surface that the icing will cover (the area of the top and the vertical faces). The depth of the cake is 10cm, and it is in the shape of a tree that is 20cm tall. The smallest triangle is 10cm wide and 5cm tall, and each pair of 'branches' is 5cm wider than the previous pair (so the cake is 20cm wide at its widest point). The height between branches is 5cm. The 'trunk' is a 5cm by 5cm square. Find the surface area that will be covered by icing.

2. I pave my patio (which is rectangular, with side lengths that are whole numbers of metres) with 1m × 1m square slabs. I use brown slabs for the outside edges (including the corners), and cream for the inside stones. If I use the same number of brown slabs as cream, how big is my patio? (You should find all possibilities for the size of the patio.)

3. Let $\binom{n}{r}$ denote the number of ways of choosing r objects from n (see the November solutions for more details). Show that $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$.

   Can you do this question without using the formula (involving factorials) for $\binom{n}{r}$?

4. A *palindromic number* is one that reads the same when its digits are reversed, such as 2332. What is the largest palindromic six-digit number that is exactly divisible by 6?

5. How many zeros does 1000000! have at the end?

6. Find all ordered triples (x, y, z) of real numbers that satisfy the following system of equations:

$$\begin{aligned} xy &= z - x - y \\ yz &= x - y - z \\ zx &= y - x - z \end{aligned}$$

   (*Ordered* triples means that we count $(1, 2, 3)$ as different from $(2, 1, 3)$ etc.: the order matters.)

7. The right-hand diagram below shows an equilateral triangle inside a square. Prove that area A + area B = area C.

8. Prove that for any six consecutive positive integers there is a prime number that divides exactly one of these integers.