BMOS Mentoring Scheme 2013 - 14 (Intermediate Level)
Sheet 2 - Example Solutions and Comments

Use Euclid's Algorithm to find the highest common factor of 30073 and 83143. (Look at the solutions to the October sheet if you don't know about Euclid's Algorithm.)
If you have time, try this question using prime factorisations. Which method is easier?

Answer: 1769.

We use Euclid's algorithm as described in the October solutions.

$83143 = 2 \times 30073 + 22997$

$30073 = 1 \times 22997 + 7076$

$22997 = 3 \times 7076 + 1769$

$7076 = 4 \times 1769$

so the highest common factor of 30073 and 83143 is 1769.

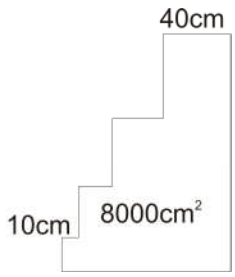
We can use this to help us find the prime factorisation. Experimentation with different possible factors shows that $1769 = 29 \times 61$. So

$30073 = 17 \times 1769 = 17 \times 29 \times 61$

$83143 = 47 \times 1769 = 47 \times 29 \times 61$.

But it would have been more difficult to find these prime factorisations without knowing that 1769 is a factor, because the primes are relatively large. Hopefully you can see that it would have been a lot harder if the primes were much bigger, whereas Euclid's algorithm would just take more steps.

2. I have built a tombola stall as shown in the diagram.



Each step is 10cm wider than the next step down, and each step is 10cm higher than the next step down. The lowest step is 10cm wide. If the area of the side is 8000cm^2, what is the height of the lowest step?

Answer: 15cm.

Let h denote the height of the lowest step in cm.

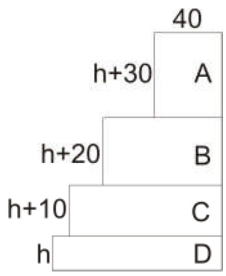
A is a rectangle with area $40(h + 30)\text{cm}^2$.

B is a rectangle with area $70(h + 20)\text{cm}^2$.

C is a rectangle with area $90(h + 10)\text{cm}^2$.

D is a rectangle with area $100h\text{cm}^2$.

(I have omitted the units from this diagram to save space.)



So $40(h + 30) + 70(h + 20) + 90(h + 10) + 100h = 8000$,

i.e., $40h + 1200 + 70h + 1400 + 90h + 900 + 100h = 8000$,

so $3500 + 300h = 8000$,

so $300h = 4500$,

so $h = 15\text{cm}$.

Of course, we could have divided up the shape differently, for example by adding vertical lines rather than horizontal. You might like to try this, to check that you still get the same answer.

3. Find all real numbers x such that $\sqrt{-3+4x} - \sqrt{13-4x} = 2$.

Answer: $x = 3$.

We shall use a similar technique to the one in the October solutions: we shall square the equation (twice, in fact) and check the values at the end.

$\sqrt{-3+4x} - \sqrt{13-4x} = 2$

Rearrange: $\sqrt{-3+4x} = 2 + \sqrt{13-4x}$

Square: $-3 + 4x = (2 + \sqrt{13-4x})^2 = 4 + 4\sqrt{13-4x} + 13 - 4x = 17 - 4x + 4\sqrt{13-4x}$

Rearrange: $4\sqrt{13-4x} = 8x - 20$

Divide by 4: $\sqrt{13-4x} = 2x - 5$

Square: $13 - 4x = (2x - 5)^2 = 4x^2 - 20x + 25$

Rearrange: $4x^2 - 16x + 12 = 0$

Divide by 4: $x^2 - 4x + 3 = 0$

Factorise: $(x - 1)(x - 3) = 0$

so $x = 1$ or $x = 3$.

Substitute $x = 1$ into the left-hand side of the original equation:

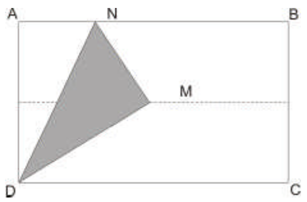
$\sqrt{-3+4x} - \sqrt{13-4x} = \sqrt{-3+4} - \sqrt{13-4} = 1 - 3 = -2$, so $x \neq 1$.

Substitute $x = 3$:

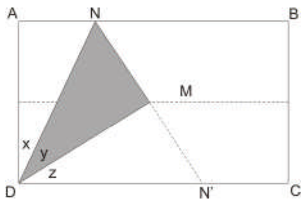
$\sqrt{-3+4x} - \sqrt{13-4x} = \sqrt{-3+12} - \sqrt{13-12} = 3 - 1 = 2$.

So the only solution is $x = 3$.

4. A rectangular piece of paper has a line drawn down the middle. One corner is then folded along DN so that the corner A coincides with a point M on the mid-line, as shown. Prove that $\angle ADN = 30°$.



Let x, y and z be the angles as marked in the diagram.



Then $x = y$, since triangle DNM is the reflection of triangle DNA in the line DN.

Reflect triangle DNM in the line DM. Since M is on the mid-line and since $\angle DMN = 90°$, when the triangle is reflected the image N' of N is on the line CD (we have a straight line NN' through M with twice the length of NM). So $y = z$. So $3x = 90°$, so $x = 30°$.

5. How many four-digit numbers have precisely three different digits, such as 2005?

Answer: 3888.

A little thought shows that it will be a good idea to count numbers with 0 as a digit separately from those that do not, since our four-digit number must not start with a 0. We shall consider three types of number separately; all are four-digit numbers with precisely three different digits. You might find it useful to have solved (or read the solution to) Question 8 before reading this solution, as some of the same ideas are used.

Type 1: Numbers that contain no 0. There are $\frac{9\times8\times7}{3\times2\times1} = 84$ possible triples of digits to appear in the number. Now suppose that our four-digit number uses the digits a, b and c. There are 3 possibilities for the repeated digit. Suppose that it is a. There are $4!/2 = 12$ ways of arranging the letters a, a, b, c. So there are $3 \times 12 = 36$ four-digit numbers using all of the digits a, b and c, so there are $36 \times 84 = 3024$ Type 1 numbers.

Type 2: Numbers that contain exactly one 0. There are $9 \times 8/2 = 36$ possible pairs that give the other distinct digits. There are 2 possibilities for the repeated digit. Suppose the digits are a, a, b, 0. There are $4!/2 = 12$ ways to arrange these, but 3 begin with 0, so we only count 9. Hence there are $36 \times 2 \times 9 = 648$ Type 2 numbers.

Type 3: Numbers that contain exactly two 0s. There are $9 \times 8/2 = 36$ possible pairs that give the other digits. There are $4!/2 = 12$ ways to arrange these, but 6 begin with a 0, so we only count 6. So there are $36 \times 6 = 216$ such numbers.

Hence in total there are $3024 + 648 + 216 = 3888$ four-digit numbers that have precisely three different digits.

(If you want some more practice with this sort of argument, you could work out the numbers of four-digit numbers that have precisely 4, 2 and 1 different digits, and check that these (together with 3888 from above) sum to the number of four-digit numbers.)

6. Show that, for all real numbers x, y, z, we have $x(x-y)+y(y-z)+z(z-x) \geq 0$. When does equality occur?

Since squares are non-negative, for all real numbers x, y, z we have $(x-y)^2+(y-z)^2+(z-x)^2 \geq 0$.

Expanding, $x^2 - 2xy + y^2 + y^2 - 2yz + z^2 + z^2 - 2zx + x^2 = 2x^2 - 2xy + 2y^2 - 2yz + 2z^2 - 2zx \geq 0$.

So $x^2 - xy + y^2 - yz + z^2 - zx \geq 0$.

So $x(x-y) + y(y-z) + z(z-x) \geq 0$.

The inequality came from the result about squares, and $\alpha^2 = 0$ if and only if $\alpha = 0$, so we have equality if and only if $x = y$, $y = z$ and $z = x$, i.e., if and only if $x = y = z$.

7. Find all positive integers (whole numbers) n such that $27n + 37$ is divisible by $3n + 1$.

Answer: $n = 1$, 2, 9.

We have that $27n + 37 = 9(3n + 1) + 28$, so $28 = (27n + 37) - 9(3n + 1)$.

So $3n + 1$ divides $27n + 37$ if and only if $3n + 1$ divides 28.

$3n + 1$ is positive, so must be a positive factor of 28.

The factors of 28 are 1, 2, 4, 7, 14, 28. We check each of these in turn:

$3n + 1 = 1$: $n = 0$ — but n must be positive.

$3n + 1 = 2$: n not an integer.

$3n + 1 = 4$: $n = 1$ is a solution (64 is divisible by 4).

$3n + 1 = 7$: $n = 2$ is a solution (91 is divisible by 7).

$3n + 1 = 14$: n not an integer.

$3n + 1 = 28$: $n = 9$ is a solution (280 is divisible by 28).

So the only solutions are $n = 1$, 2, 9.

8. I have to select four of the seven dwarves to play a game of bridge. In how many ways could this be done? (The order in which the dwarves are chosen does not matter.) Can you find a formula for the number of ways in which r dwarves can be chosen from a group of n dwarves?

Answer: 35 ways for the bridge team; $\frac{n!}{r!(n-r)!}$ for the general problem.

There are

7 possibilities for the first dwarf,

6 possibilities for the second dwarf,

5 possibilities for the third dwarf, and

4 possibilities for the fourth dwarf,

which gives $7 \times 6 \times 5 \times 4 = 840$. But this counts $ABCD$ as different from $BCDA$, and $BACD$, and so on. There are $4 \times 3 \times 2 \times 1 = 24$ ways of arranging the four chosen dwarves, so there are $840/24 = 35$ ways of choosing the bridge team.

We can use this method to count the number of ways in which r dwarves can be chosen from a group of n.

There are

n possibilities for the first dwarf,

$n - 1$ for the second dwarf,

$n - 2$ for the third dwarf,

$\ldots$, and

$n - r + 1$ for the r^{th} dwarf,

which gives $n(n-1)\ldots(n-r+1)$. As before, we must now divide by the number of ways of arranging each group of r dwarves, namely $r!$ (see the October sheet for details of this).

Hence there are $\frac{n(n-1)\ldots(n-r+1)}{r!}$ ways.

Note that

$$n(n-1)\ldots(n-r+1) = \frac{n(n-1)\ldots(n-r+1)(n-r)(n-r-1)\ldots(2)(1)}{(n-r)(n-r-1)\ldots(2)(1)} = \frac{n!}{(n-r)!},$$

so there are $\frac{n!}{r!(n-r)!}$ ways.

We have a special notation for the number of ways of choosing r objects from n (where order doesn't matter): we write

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

This is also sometimes written as ${}^nC_r$, and is read as "n choose r".